LECTURE 4

1 Why do people save?

Consider a consumer with a two-period lifetime who can borrow and lend at the riskless rate r. The consumer maximizes

$$\max_{c_1,c_2} u(c_1) + \frac{1}{1+\rho} u(c_2) \tag{1}$$

$$\text{s.t. } a_{t+1} = (1+r)\, a_t + (y_t - c_t) \tag{2}$$

$$a_1 = 0 \tag{3}$$

$$a_3 \geq 0, \tag{4}$$

with $\rho > 0$, u strictly increasing and concave in c_t and $u_c(0) = \infty$. a_t is the stock of wealth at the beginning of period t while y_t and c_t are respectively labour income and consumption at the end of period t.

- The *dynamic constraint* (2) is not really a constraint. It is an accounting identity (always holds).

- Constraint (2) is still satisfied even if the consumer borrows an infinite amount to enjoy an infinite consumption flow. For the problem to be well defined we have to rule this out $\rightarrow$ *solvency constraint* (3). Individual cannot die with negative stock of assets.

- An alternative and useful way of imposing the solvency constraint is to sum the dynamic constraint forward and impose solvency to obtain

$$c_1 + \frac{c_2}{1+r} \leq y_1 + \frac{y_2}{1+r}. \tag{5}$$

  This is the *intertemporal budget constraint* and is fully equivalent to the solvency constraint (3).

Given non-satiation, optimality requires the solvency or intertemporal budget constraints to hold as equalities. $a_3 = 0$. We can then replace for c_2 in (1) using (5) as an equality to obtain

$$\max_{c_1} u\left(c_1\right) + \frac{1}{1+\rho} u\left(y_2 + \left(1+r\right)\left(y_1 - c_1\right)\right), \tag{6}$$

with FOC

$$u_c(c_1) = \frac{(1+r)}{1+\rho} u_c(c_2). \tag{7}$$

This is the intertemporal optimality condition (aka **Euler equation**) the consumer problem. The expected marginal utility of consumption discounted at the market interest rate has to be equalized across the two periods.

- **Step 1:** The Euler equation is a statement on the *slope* of expected marginal utility over time not its level.
- **Step 2:** To recover the *level* of consumption we need to replace in the intertemporal budget constraint.

The above example usefully highlights the main motives for saving.

1. Consumption tilting motive. The Euler equation implies that the consumption profile is upward sloping if $\frac{(1+r)}{1+\rho} > 1$; i.e. if individuals subjective discount rate is lower than the market rate of return on saving. Viceversa if $\frac{(1+r)}{1+\rho} < 1$. Even if $y_1 = y_2$ individuals save in the first case and borrow in the second one.

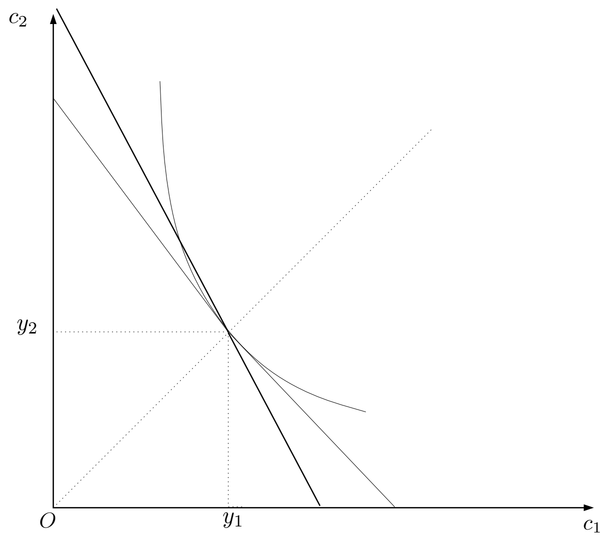

Figure 1: Consumption tilting

2. Consumption smoothing motive. Assume $\frac{1+r}{1+\rho} = 1$ (no consumption tilting motive). The Euler equation requires

$$c_1 = c_2. \tag{8}$$

Agents want a flat consumption profile, because of decreasing marginal utility. They will save (borrow) in the first period if $y_1 > y_2$ ($y_1 < y_2$).

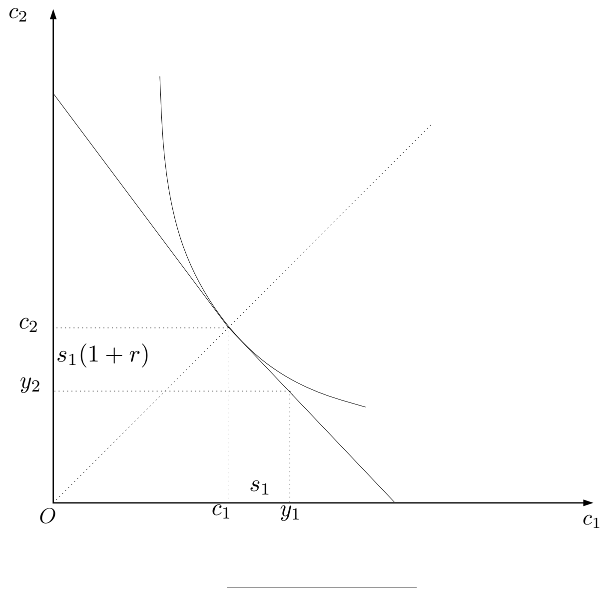

Figure 2: Consumption smoothing

Main insight of the intertemporal theory of consumption. If allowed to borrow and lend agents with strictly concave preference will use saving as a buffer stock to smooth consumption in the face of a non-flat or non-smooth income profile.

Example 1: Consider an increase in first-period income Δy_1. Agents will increase both c_1 and saving to spread the increase in income across the two periods. Consumption goes up by less than income.

Example 2: Consider a permanent increase in income $\Delta y_1 = \Delta y_2$. Consumption goes up by roughly the full amount of the income increase and saving is little affected. The same insight applies to the case in which the non-smoothness in income is due to (a) income uncertainty or (b) finite working life (retirement).

2 Ramsey-Cass-Koopmans model

- Relax the assumption that saving rate is *exogenous* and *constant* .
  - Saving rates increase with economic development.
  - Richer dynamics of savings imply richer transitional dynamics.
- Market economy (Households and firms, competitive factor and product markets).

2.1 Households

- There are H households in the economy. Households members live forever and new members are born at rate n; i. e. at time t each household has size $L_t = e^{nt}$ (L_0 is normalized to 1). This is also the household inelastic labour supply.
- Each household maximizes the *discounted* sum of utilities of present and future dinasties; i.e.

$$U_0 = \int_0^{\infty} u(C_t) L_t e^{-\rho t} dt, \tag{9}$$

where C_t is consumption per head. The weight of each generation depends on:

1. Its size L_t.
2. Its distance into the future (discounting).

- The instantaneous utility function is parameterized by

$$u(c_{t)}) = \frac{C_t^{1-\theta}}{1-\theta}, \tag{10}$$

with $\theta > 0$. Constant relative risk aversion

$$-\frac{u''(C)C}{u'(C)} = \theta. \tag{11}$$

When $\theta \to 1$, $u(C) \to \log C$.

- Household *dynamic budget constraint*

$$\dot{B}_t = r_t B_t + W_t L_t - C_t L_t. \tag{12}$$

B_t is the total stock of assets in each household. Two types of assets: ownership claims on physical capital and bonds. They are perfect substitutes (both riskless), hence they yield same rate of return r_t. W_t is the real wage. Agents can freely borrow/lend at rate r.

- Competitive markets. Households take path for ${}^{\infty}_{t=0}\{r_t, W_t\}$ as given.
- Technological progress: stock of knowledge $A_t = e^{gt}$. (A_0 is normalized to 1).
- Solvency constraint: households cannot run a pyramidal scheme (aka Ponzi game) by borrowing to rollover debt forever. Denote by

$$R_s = \int_{\tau=0}^{s} r_\tau d\tau, \tag{13}$$

the compounded interest factor[1] between time 0 and s. The inability to run a pyramidal scheme then implies

$$\lim_{t\to\infty} B_t e^{-R_t} \geq 0. \tag{14}$$

- **Useful result:** $\partial R_t/\partial t = r_t$

[1] $R_s = rs$ if r_t is constant

Denote by small letters variables in efficiency units of labour. $b_t = B_t/(A_t L_t)$, $c_t = C_t/A_t$ and $w_t = W_t/A_t$ (note that C_t and W_t were already per capita).

Using the utility function and variables in per capita terms (9) becomes

$$U_0 = \int_0^\infty \frac{c_t^{1-\theta}}{1-\theta} e^{-[\rho-(1-\theta)g-n]t} dt \tag{15}$$

For the present value of lifetime utility to be bounded the effective discount rate must be negative or

$$\rho - (1-\theta)g - n > 0. \tag{16}$$

Dividing both side by B_t the dynamic budget constraint (12) becomes

$$\frac{\dot{B}_t}{B_t} = r_t + \frac{W_t L_t}{B_t} - \frac{C_t L_t}{B_t}. \tag{17}$$

Using the properties of logarithms the latter can be written as

$$\frac{\dot{b}_t}{b_t} + (n+g) = r_t + \frac{w_t}{b_t} - \frac{c_t}{b_t} \tag{18}$$

or

$$\dot{b}_t - (r_t - n - g) b_t = w_t - c_t. \tag{19}$$

The solvency constraint can also be rewritten in per capita terms as

$$\lim_{t\to\infty} b_t e^{-[R_t-(n+g)t]} \geq 0. \tag{20}$$

To obtain the household intertemporal budget constraint we need to integrate the dynamic constraint and *impose* the solvency constraint.

Multiply both side of (19) by $e^{-R_t+(n+g)t}$. Notice that it is

$$\left(\dot{b}_t - (r_t - n - g)b_t\right) e^{-R_t+(n+g)t} = \frac{d(b_t e^{-R_t+(n+g)t})}{dt}. \tag{21}$$

Take integrals of both side of (19) between 0 and ∞

$$\int_0^\infty \frac{d(b_t e^{-R_t+(n+g)t})}{dt} dt = \int_0^\infty (w_t - c_t) e^{-R_t+(n+g)t} dt. \tag{22}$$

This can be rewritten as

$$\lim_{t\to\infty} b_t e^{-R_t+(n+g)t} - b_0 = \int_0^\infty (w_t - c_t) e^{-R_t+(n+g)t} dt. \tag{23}$$

Imposing the solvency constraint (20) we then obtain the household *intertemporal budget constraint*

$$\int_0^\infty c_t e^{-R_t+(n+g)t} dt \leq b_0 + \int_0^\infty w_t e^{-R_t+(n+g)t} dt. \tag{24}$$

The present discounted value of lifetime household consumption cannot exceed the present discounted value of their lifetime wealth.

Note that (20) and (24) both represent the household intertemporal budget constraint and can be used interchangeably.

We can no solve the household optimization problem which consists of choosing a path for c_t that maximizes (15) subject to (24). The associated Lagrangean is

$$\mathcal{L} = \int_0^\infty \frac{c_t^{1-\theta}}{1-\theta} e^{-[\rho-(1-\theta)g-n]t} dt + \lambda \left[b_0 + \int_0^\infty (w_t - c_t) e^{-R_t+(n+g)t} dt \right]. \tag{25}$$

The sequence of FOCs, one for each t, is

$$c^{-\theta} e^{-[\rho-(1-\theta)g-n]t} = \lambda e^{-R_t+g+nt}, \tag{26}$$

which simplifies to

$$c^{-\theta} = \lambda e^{-[R_t-(\rho+\theta g)t]}. \tag{27}$$

Taking logs and time derivatives the FOC can be rewritten as

$$\frac{\dot{c}}{c} = \frac{r_t - \rho - \theta g}{\theta}. \tag{28}$$

This is the Euler equation in continuous time. It gives the rate of change of consumption along an optimal path.

Note:

$$\frac{\dot{C}}{C} = \frac{\dot{c}}{c} + g = \frac{r_t - \rho}{\theta}. \tag{29}$$

Consumption per head (the choice variable) grows if rate of interest exceeds subjective rate of discount (consumption tilting).

2.2 Firms

- Standard CRS production function with labour augmenting technological progress. $Y = F(K, AL)$. Given CRS, nothing is lost by normalizing H to 1.
- Capital depreciation rate δ.
- Firms choose K_t, L_t to maximize profits

$$\Pi_t = F(K_t, A_tL_t) - W_tL_t - (r_t + \delta)K_t = A_tL_t\left[f(k) - w_t - (r_t + \delta)k_t\right] \tag{30}$$

FOCs for k_t is

$$f'(k_t) = r_t + \delta. \tag{31}$$

Zero profit condition (because of CRS) implies

$$w_t = f(k_t) - f'(k_t)k_t. \tag{32}$$

The wage a worker receives satisfies

$$W_t = A_t[f(k_t) - f'(k_t)k_t]. \tag{33}$$

2.3 Equilibrium

An equilibrium is a sequence of vectors ${}_{t=0}^{\infty}\{c_t, b_t, k_t, L_t^d, L_t^s, w_t, r_t\}$ such that

1. The Euler equation (28), the dynamic constraint (19) are satisfied *and* the intertemporal budget constraint (24) (or equivalently the solvency constraint (20)) is satisfied.

2. The labour market clears: labour supply $L_t^s = HL_t$ equals labour demand (indeterminate).

3. The capital market clears: $b_t = k_t$

4. Factor prices satisfy (31) and (32).

For the time being we disregard the solvency constraint. Replace for factor prices w_t, r_t and the equilibrium value of b_t in the dynamic constraint to obtain

$$\dot{k}_t = f(k_t) - c_t - (\delta + n + g)k_t. \tag{34}$$

This together with the Euler equation

$$\frac{\dot{c}_t}{c_t} = \frac{f'(k_t) - \delta - \rho - \theta g}{\theta}. \tag{35}$$

forms a system of two differential equations in the two variables k_t and c_t.

2.3.1 Steady state equilibrium

The Euler equation (40) implies that for c_t to grow at a constant rate k_t has to be constant or $\dot{k}_t = 0$. It follows from (39) that

$$f(k^*) - c^* = (\delta + n + g)k^*. \tag{36}$$

Therefore also c is constant and the Euler equation becomes

$$f'(k^*) = \delta + \rho + \theta g. \tag{37}$$

- Saving equals replacement investment, as in the Solow growth model. But in Solow saving rate affects steady state capital stock. Here the causation is reversed. The capital stock is fully determined by the exogenous steady state rate of return on capital (i.e. δ, ρ, θ and g.) In Ramsey, it is the saving rate which adjusts.

  Suppose the production function is Cobb-Douglas. Compare (37) with

$$f'(k^*) = \alpha \frac{f(k^*)}{k^*} = \frac{\delta + g + n}{s}. \tag{38}$$

  The rate of population growth n does not affect the steady state capital stock in the Ramsey model (intuition it affects saving and replacement investment by the same amount). It only results in a higher saving rate.

- Capital stock below its golden rule level. Golden rule satisfies $f'(k^{GR}) = \delta+n+g$ while steady state level satisfies (37). The condition for boundedness of lifetime utility (16)implies $f'(k^*) > f'(k^{GR})$ or $k^* < k^{GR}$.

- Check that the steady state equilibrium satisfies the solvency constraint (Hint: use the solvency constraint (20)).

2.3.2 Off-steady state equilibrium (transitional dynamics)

Everything is in the book, but two important things to remember.

- k(0) given but c(0) is endogenous.

- Off the steady state the interest rate adjusts to generate the saving in line with capital demand. This generates the consumption tilting desire that provides the necessary saving.

$$\dot{k}_t = f(k_t) - c_t - (n+g)k_t. \tag{39}$$

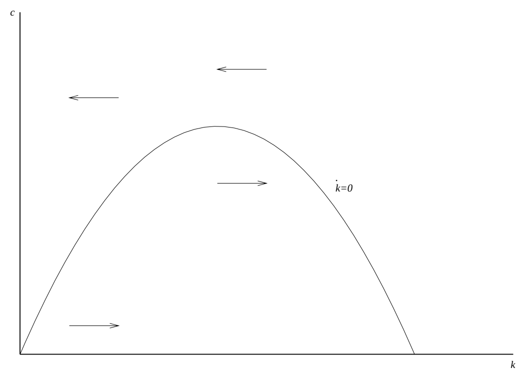

Figure 3: Dynamics of k

and

$$\frac{\dot{c}}{c} = \frac{f'(k_t) - \rho - \theta g}{\theta}. \tag{40}$$

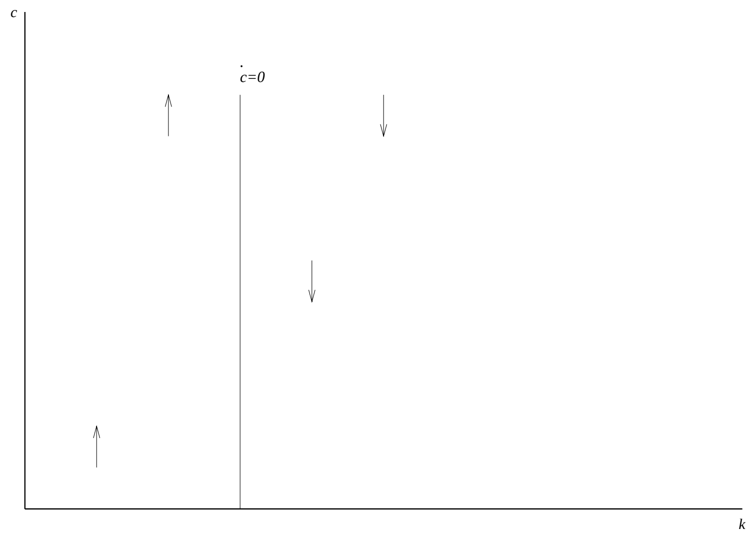

Figure 4: Dynamics of c

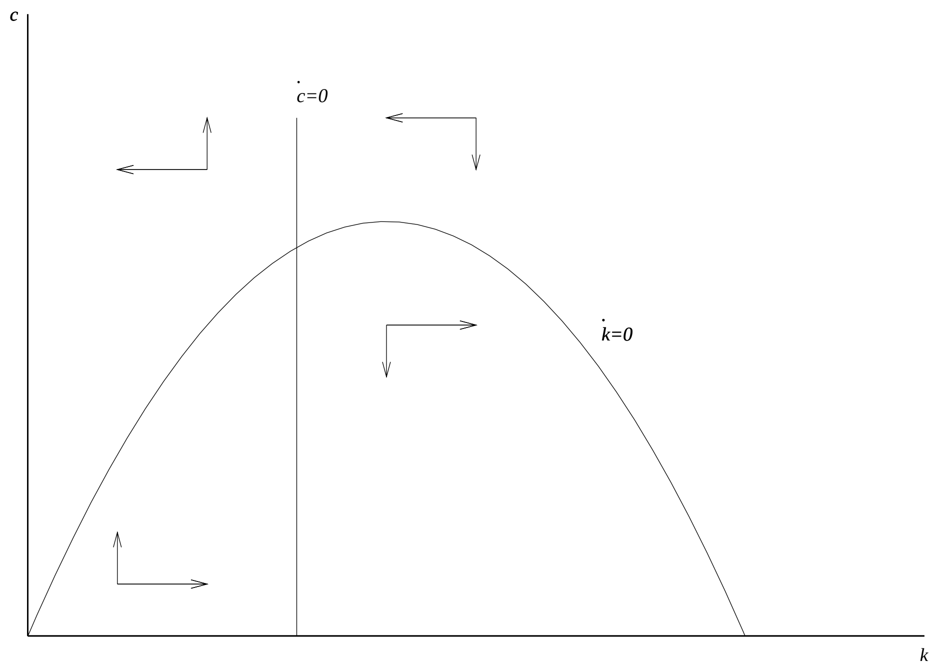

Figure 5: Joint dynamics of k and c

- Equilibrium vector satisfies a system of **two** non-linear differential equations. Hence the time path of variables is differentiable.

- Many integrals from one derivative$\rightarrow$ Many paths satisfying the above system. Need **two** boundary conditions (one for each equation). k_0 given is one. Given k_0 what pins down c_0? Solvency constraint is another one.

- Paths above the saddle path intersect the vertical axis in finite time. At that point there is no capital to uninstall and $c = f(0) = 0 \rightarrow c$ jumps down violating differentiability (Euler equation).

- Below the saddle path eventually $f'(k) < g+n$ (the economy is to the right of Golden rule)and $\lim_{t\rightarrow\infty} k_t e^{(-R(t)+(n+g)t)} = \infty$. The economy eventually ends with negative c, which cannot be optimal as it violates the Inada condition.

- Saddle path is the unique time path which satisfies all requirements. c_0 jumps on the saddle path.

- Path of interest rates and growth in income per capita off steady state.

- Shocks to parameters. Causation changed, but model's response to changes in saving rate brought about by changes in ρ as in the Solow model.
- Higher n reduces consumption and increases saving, but has no effect on capital and output in intensive units.

2.4 Alternative environments

1. Household-producer who maximizes (15) subject to the accumulation constraint (39).
2. Social planner who maximizes the utility (15) of the representative agent subject to the feasibility constraint (39). The first welfare theorem (competitive equilibrium is efficient if: no externalities, concave production function, complete markets, finite number of agents (households)) implies that the equilibrium coincides with the (efficient) decentralized one.