Macroeconomics A

Solution to problem set 4

1. The capital accumulation equation is

$$\frac{\dot{K}}{K} = sK^{\alpha-1}(AL)^{1-\alpha} = s\tilde{k}^{\alpha-1}. \tag{1}$$

Knowledge accumulates according to

$$\frac{\dot{A}}{A} = BK^{\alpha}A^{-\alpha}L^{1-\alpha} = B\tilde{k}^{\alpha}L. \tag{2}$$

For the two rates of growth to be constant $\tilde{k}$ has to be constant (remember that L does not grow). This implies that in steady state it is

$$\frac{\dot{K}}{K} = \frac{\dot{A}}{A}. \tag{3}$$

Equating (1) and (2) we obtain

$$s\tilde{k}^{\alpha-1} = B\tilde{k}^{a}L \tag{4}$$

which can be solved for the steady state level of $\tilde{k}$

$$\tilde{k^*} = \frac{s}{BL}. \tag{5}$$

Replacing in (2) gives

$$\frac{\dot{A}}{A} = B\left(\frac{s}{BL}\right)^{\alpha} L = s^{\alpha}(BL)^{1-\alpha}. \tag{6}$$

This is also the steady state rate of growth of K and Y. It is increasing in s and B. The intuition is that a higher s increases capital accumulation for given strength of the externality B. A higher B increases the externality for given s.

2. Consider a consumer living for two periods. Her utility function is

$$\log(c_1) + \beta\log(c_2) \tag{7}$$

where c_1 and c_2 are respectively consumption in the first and second period and $0 < \beta < 1$ is the factor at which she discounts future utility. The consumer is born with no assets, receives labour income y_1 and y_2 respectively in period 1 and 2 and can freely borrow and lend at the market interest rate r.

(a) The consumer dynamic budget constraint is

$$a_{t+1} = (1+r)a_t + y_t - c_t. \tag{8}$$

(b) With a finite lifetime solvency requires that the consumer does not die in debt. Given that the lifetime lasts 2 periods here, it is

$$a_3 \geq 0. \tag{9}$$

Summing the dynamic budget constraint over time gives

$$a_3 = (1+r)(y_1 - c_1) + y_2 - c_2. \tag{10}$$

Imposing the solvency constraint (9) yields the IBC

$$(1+r)(y_1 - c_1) + y_2 - c_2 \geq 0. \tag{11}$$

(c) The Euler equation can be obtained by noticing that given that the marginal utility of consumption is always positive the IBC (11) is satisfied as an equality; i.e.

$$(1+r)(y_1 - c_1) + y_2 - c_2 = 0. \tag{12}$$

Using it to replace for c_2 in the utility function and maximizing yields the Euler equation

$$\frac{1}{c_1} = \beta(1+r)\frac{1}{c_2}. \tag{13}$$

It implies that $c_2/c_1 > 1$ if $\beta(1+r) > 1$. The consumption profile is upward sloping if the market rate of interest is high relative to the subjective discount rate.

(d) The system formed by the Euler equation (13) and the IBC (12) is a system of two equations in the two unknowns c_1 and c_2. Solving for c_1 yields

$$c_1 = \frac{1}{1+\beta}\left(y_1 + \frac{y_2}{1+r}\right). \tag{14}$$

First period saving is then

$$s_1 = ra_1 + y_1 - c_1 = y_1 - c_1 = \frac{1}{1+\beta}\left(\beta y_1 - \frac{y_2}{1+r}\right). \tag{15}$$