ECOM 009 Macroeconomics B

Lecture 4

Giulio Fella

Where do we stand

- We have considered throughout the self-insurance problem when the only asset available is the risk-free asset.
- We have seen how prudence - $u''' > 0$ is sufficient to generate a precautionary saving motive.
  - The introduction of a precautionary saving motive is able to account for a number of empirical failures of the PICH.
  - Two problems still persist even after the introduction of precautionary saving if the utility function is negative exponential.

Linear consumption function with exponential utility

Remember the consumption and saving functions when utility is negative exponential and income an AR(1) process.

$$
\begin{aligned}
c_t &= -\frac{\gamma r \sigma^2}{2(1-\lambda+r)^2} + r a_t + \frac{r}{1-\lambda+r} y_t \\
s_t &= \frac{\gamma r \sigma^2}{2(1-\lambda+r)^2} + \frac{1-\lambda}{1-\lambda+r} y_t \\
c_{t+1} - c_t &= r(a_{t+1} - a_t) + \frac{r}{1-\lambda+r}(y_{t+1} - y_t) \\
&= \frac{\gamma r^2 \sigma^2}{2(1-\lambda+r)^2} + \frac{r(1-\lambda)}{1-\lambda+r} y_t + \frac{r}{1-\lambda+r}(y_{t+1} - y_t) \\
&= \frac{\gamma r^2 \sigma^2}{2(1-\lambda+r)^2} + \frac{r}{1-\lambda+r} \epsilon_{t+1}
\end{aligned}
$$

Problems when consumption is linear

Too low MPC out of wealth windfalls

- The marginal propensity to consume out of wealth windfall (MPC_a) is the same as under certainty and under quadratic utility.
  For a generic income process, we have.
  - Exponential utility:

$$c_t = -K + r(a_t + H_t)$$

  - Certainty and quadratic utility:

$$c_t = r(a_t + H_t).$$

- In both case $MPC_a = r \sim 0.03$ while it is roughly 0.3 in the data.

Problems when consumption is linear

Consumption and wealth are random walks

- Exponential utility implies

$$E_t c_{t+1} = c_t + K \tag{78}$$

and

$$a_{t+1} = a_t + \frac{K}{r} + H_t - E_t H_{t+1}. \tag{79}$$

- Consumption and wealth are random walks *with drift.*
- Both are expected to grow without bound independently of wealth level. Therefore saving is independent of wealth

Why it is unpalatable

Consider, for simplicity, the case $H_t - E_t H_{t+1} = 0$ (no consumption smoothing).

- As long as income is uncertain, it is $K > 0$ and individuals keep postponing consumption however rich they are.
- **But...**, infinite wealth should allow the individual to perfectly self-insure and greatly reduce the precautionary saving motive.
  - Recall:
$$E_t[c_{t+1} - c_t] = -\frac{u'''(c_t)}{u''(c_t)} \frac{E_t[c_{t+1} - c_t]^2}{2}. \tag{80}$$
  - The last term goes to zero when $a_t \to \infty$.
- So the random walk result is very unpalatable.

Possible resolutions

- Throughout we have maintained two assumptions:
  - $\beta(1 + r) = 1$
  - Infinite horizon.
- Together they imply that the consumption and wealth processes diverge to infinity whenever there is a precautionary saving motive.
- We do not want to give up the second assumption (it is very convenient).

Plan of this lecture

1. Show that
   - if *income is uncertain* borrowing constrains are also sufficient to generate a precautionary saving motive;
   - requiring consumption to be non-negative implies a (natural) borrowing constraint;
   - if income is uncertain wealth and consumptio are bounded if and only if $\beta(1 + r) < 1$.
2. Argue that precautionary saving together with $\beta(1 + r) < 1$ imply that
   - consumers want to accumulate wealth up to a target level and decumulate it above such level; i.e. saving is decreasing in wealth;
   - the average marginal propensity to consume out of wealth windfalls is significantly larger than r.

Self-insurance with borrowing constraints

- Relevant readings: Ljungqvist and Sargent, chapter 16 (judiciously) and 17.3-17.5.
- We now tackle bulled point 1. in the previous slide.

Self-insurance with borrowing constraints

Consider the consumer maximization problem in the presence of borrowing constraints. Assume income is a first-order Markov process. Assume also it is bounded (for simplicity). That is $y_t \in [\underline{y}, \bar{y}]$ for any t.

$$W(a_t, y_t) = \max_{c_t, a_{t+1}} u(c_t) + \beta E W(a_{t+1}, y_{t+1}) \tag{81}$$

$$\text{s.t. } a_{t+1} = (1+r)a_t + y_t - c_t \tag{82}$$

$$a_t \text{ given, solvency} \tag{83}$$

$$a_{t+1} \geq -b. \tag{84}$$

The last one is a borrowing constraint as long as $b \geq 0$. Resources carried over to the next period cannot fall below some bound.

Euler equation with borrowing constraints

The optimality condition for the above problem is the following

$$u'(c_t) \geq \beta(1+r)E_t u'(c_{t+1}), \qquad = \text{ if } a_{t+1} > -b. \tag{85}$$

- The standard Euler equation holds only if the borrowing constrained is slack.
- If not, it is $a_{t+1} = -b$. From the dynamic constraint, consumption is given by

$$c_t = (1+r)a_t + y_t + b. \tag{86}$$

  - The individual consumes everything, including her maximum borrowing allowance b.
  - She would like to borrow more to equate marginal utility today and tomorrow.

Steady-state consumption without income uncertainty

- If income is deterministic
  - as long as $\beta(1+r) \leq 1$ consumption converges to a finite limit as $t \to \infty$ (we know this from Ramsey model);
  - consumption diverges if $\beta(1+r) > 1$.
- We now compare this result to the case in which income is uncertain.

Steady-state consumption with income uncertainty

Define the **non-negative** r.v.

$$M_t = [\beta(1+r)]^t u'(c_t) \geq 0. \tag{87}$$

- It is

$$M_{t+1} - M_t = [\beta(1+r)]^t [\beta(1+r)u'(c_{t+1}) - u'(c_t)]. \tag{88}$$

- We can rewrite equation (85) as

$$E_t(M_{t+1} - M_t) \leq 0. \tag{89}$$

- This implies that M_t is a (non-negative) supermartingale.

Supermartingale convergence theorem

Theorem

Let M_t be a non-negative supermartingale. For $t \to \infty$, M_t converges almost surely to a non-negative random variable $\bar{M}$ with $E(\bar{M})$ finite.

$\beta(1+r) \geq 1$

1. $\beta(1+r) > 1$.
   - $[\beta(1+r)]^t$ diverges to infinity.
   - As $M_t = \beta(1+r)]^t u'(c_t)$ converges it has to be $\lim_{t\to\infty} u'(c_t) = 0$
   - As long as $u' > 0$, c_t diverges to infinity and so a_t, as long as the PDV of labour income is finite.
2. $\beta(1+r) = 1$.
   - Same result as when $\beta(1+r) > 1$, under fairly general conditions on the income process.
   - Drastically different from the no-uncertainty case.
     - However little uncertainty there is, it implies that consumption diverges in the limit.

$\beta(1+r) < 1$

3. $u'(c_t)$ does not have to converge to zero for M_t to converge.

- Consumption can stay finite and vary with shocks.
- The average levels of consumption and assets remain finite.
- Intuition: as long as the borrowing constraint can be hit with positive probability consumers save in order to avoid such event (precautionary saving).
  - The precautionary saving motive implies consumption is expected to grow if $\beta(1+r) = 1$.
  - If $\beta(1+r) < 1$ the consumption tilting motive implies a downward sloping consumption profile in the absence of a precautionary saving motive (assets would converge to the borrowing limit).
- The two effects balance each other out and the wealth distribution is non-degenerate.

The natural borrowing limit

What happens if $c_{t+s} \geq 0$ (e.g. as under CRRA, or additional constraint)?

- The intertemporal budget constraint implies

$$a_t \geq \frac{1}{1+r} \sum_{s=0}^{\infty} \frac{c_{t+s}}{(1+r)^s} - \frac{1}{1+r} \sum_{s=0}^{\infty} \frac{y_{t+s}}{(1+r)^s} \tag{90}$$

- Given $c_{t+s} \geq 0$, the above inequality implies

$$a_t \geq -\frac{1}{1+r} \sum_{s=0}^{\infty} \frac{y_{t+s}}{(1+r)^s} = -h_t \tag{91}$$

- Ayagari (1994) calls this the *natural borrowing limit* as it follows by imposing only the *natural* constraint $c_{t+s} \geq 0$.

The natural borrowing limit

Certain vs uncertain income

The borrowing limit has to hold with probability one.

1. No labour income uncertainty.
   - The maximum amount the individual can repay at time t is the present value of her labour income from then onwards.
2. Labour income uncertainty.
   - Since, there is a positive probability that the individual receives her worst possible income realization $\underline{y}$ at all possible $t+s$, equation (91) has to hold in this worst case scenario for it to hold with probability one.
   - The *natural borrowing limit* is

$$a_t \geq -\frac{1}{1+r}\sum_{s=0}^{\infty}\frac{\underline{y}}{(1+r)^s} = -\frac{\underline{y}}{r} = -h. \tag{92}$$

Natural vs ad-hoc borrowing limit

- If consumption cannot be negative any *ad-hoc* borrowing limit b such that $b > h$ is slack at the natural borrowing limit.
- The natural borrowing limit is tighter than the ad-hoc one if $b > h$.
- The borrowing constraint associated with non-negative consumption is $a_{t+1} \geq -\phi$ with $\phi = \min\{b, h\}$

Non-negative consumption and precautionary saving

- As long as consumption cannot be negative it implies a borrowing constraint.
  - The constraint is binding with positive probability in the presence of income uncertainty.
  - Borrowing constraint are sufficient to generate a precautionary saving motive when income is uncertaint.
    - Income uncertainty generates precautionary saving *even with quadratic utility* as long as consumption cannot be negative.
- Note that the natural borrowing limit already implies solvency.
- The solvency constraint does not need to be imposed separately.

Precautionary saving and general equilibrium

- Relevant readings: Aiyagari (1994) and Ljungqvist and Sargent, chapter 17.6.
- In this section we want to study the implications of precautionary saving for the equilibrium interest rate and the equilibrium stock of capital.
- In order to deerive the aggregate supply of assets we need to derive the individual asset supply (saving function).
  - Deriving the saving function when consumption is not linear.
  - Namely, interplay of precautionary saving and consumption tilting.

Individual asset supply with no income uncertainty

Assume that y is fixed.

1. $\beta(1+r) < 1$. If the individual is not borrowing constrained the Euler equation holds and implies

$$u'(c_t) = \beta(1+r)u'(c_{t+1}) < u'(c_{t+1}). \tag{93}$$

   - As $t \to \infty$ consumption decreases until $a_t = -\phi$.
   - In the limit the individual asset supply is $-\phi$.

2. $\beta(1+r) = 1$.
   - With constant income nothing to smooth. $s_t = a_{t+1} - a_t = 0$.
   - Wealth is constant at its initial value and $c_t = y + ra_t = y + ra_0$ for any t.
   - Individual asset supply is whatever the individual had at the beginning of time.

3. $\beta(1+r) > 1$. In the limit the individual asset suppy is ∞.

Individual asset supply with income uncertainty

Assume the income process y_t is i.i.d. (i.e. $E_t y_{t+s} = \mu$ for any $s > 1$). Assume also it takes values in a finite set $y_1, y_2, \ldots, y_k$ with $y_i < y_{i+1}$.
The consumer problem is

$$W(a_t, y_t) = \max_{a_{t+1}} u(y_t + (1+r)a_t - a_{t+1}) + \beta E W(a_{t+1}, y_{t+1}) \tag{94}$$

$$\text{s.t. } c_t + a_{t+1} = y_t + (1+r)a_t \tag{95}$$

$$a_t \text{ given, } a_{t+1} \geq -\phi. \tag{96}$$

Note that since y_t is i.i.d. its current realization does not help predict its future variable.

Change of state variable

All that matters for the consumer decision problem (the relevant state variable) is not a_t and y_t separately, but the total cash-at-hand $z_t = (1+r)a_t + y_t + \phi$. Therefore we can write the consumer value function as a function of z_t

$$W(z_t) = \max_{a_{t+1}} u(z_t - a_{t+1} - \phi) + \beta E W(z_{t+1}) \tag{97}$$

$$\text{s.t. } c_t + a_{t+1} = z_t - \phi \tag{98}$$

$$z_{t+1} = (1+r)a_{t+1} + y_{t+1} + \phi \tag{99}$$

$$a_t \text{ given, } a_{t+1} \geq -\phi. \tag{100}$$

If we denote by $\hat{a}_t = a_t + \phi$ finding a solution for the function a_{t+1} is equivalent to finding a solution for its translation $\hat{a}_{t+1}$. It is more convenient to work with the latter.

The saving policy

The solution for the optimal policy $\hat{a}_{t+1}$ is a function

$$\hat{a}_{t+1} = \hat{a}_{t+1}(z_t; r, \phi). \tag{101}$$

- From the previous section, if $\beta(1+r) \geq 1$ the limit of the policy function a_{t+1} as $t \to \infty$ is infinite.
- So we are interested in studying its properties only for the non-degenerate case $\beta(1+r) < 1$.
- If $\beta(1+r) < 1$ the policy functions are illustrated in Figure 9 (all figures that follow are from Aiyagari 1994). We know, from standard theorems, that it is a continuous function under fairly general conditions.

$c_t(z_t),\ \hat{a}_{t+1}(z_t)$ when $\beta(1+r) < 1$

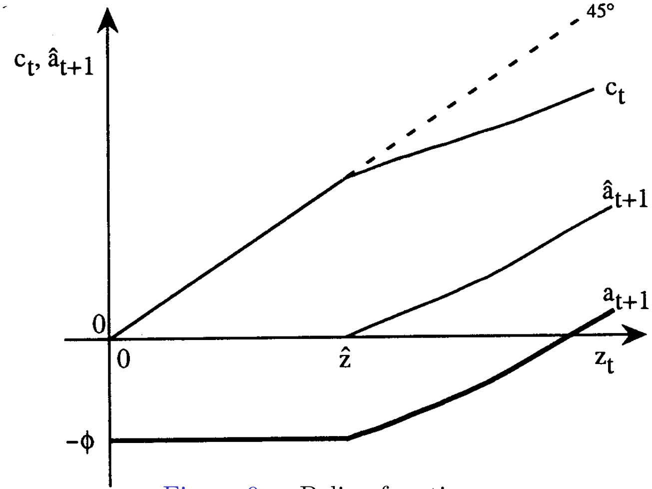

Figure 9 : Policy function

Saving vs cash at hand

- The consumer is borrowing constrained for low values of cash at hand.
  - Below some cutoff value $\hat{z}$, it is $\hat{a}_{t+1} = 0$ (i.e. $a_{t+1} = -\phi$) and $c_t = z_t$.
  - Consumption increases one-for-one with cash-at-hand while future wealth is constant.
- For $z_t > \hat{z}$ the consumer is not borrowing constrained.
  - and the Euler equation demands that she divides any increase in cash-at-hand today between consumption today and tomorrow;
  - a_{t+1} is increasing in z_t at a rate positive but strictly less than one and consumption increases at the complementary rate.

The consumption function is concave

- With borrowing constraints the consumption function is concave (at least globally).
- The marginal propensity to consume out of cash at hand is one if the individual is borrowing constrained and decreases in wealth to converge asymptotically to roughly r. This is consistent with
  - Keynes' insight that richer consumers save more;
  - an average (across consumers)marginal propensity to consume out of wealth windfalls in line with the data.
- When $z_t \to \infty$, $\hat{a}_{t+1}$ and c_t increase in z_t at the same rate as they would under risk neutrality.

A useful alternative representation: $z_{t+1}(z_t)$

Using the definition of cash-at-hand we can write

$$z_{t+1} = (1+r)(\hat{a}_{t+1} - \phi) + y_{t+1} + \phi = (1+r)\hat{a}_{t+1}(z_t; r, \phi) + y_{t+1} - r\phi,$$

which gives tomorrow cash-at-hand as a function of today's.

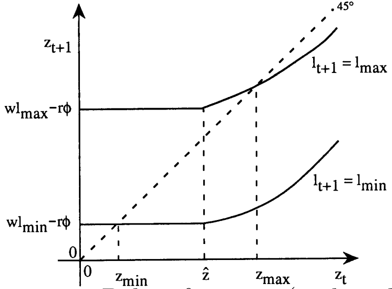

Figure 10 : Policy function (cash at hand)

Intuition

The figure contains the main insight of precautionary saving when wealth is bounded (i.e. $\beta(1+r) < 1$).

- Mapping from z_t to z_{t+1} for the two extreme value of y_{t+1}, $\underline{y} = wl_{min}$ and $\bar{y} = wl_{max}$.
- Mapping is stochastic as y_{t+1} is stochastic as of time t. Given z_t, y_{t+1} determines the appropriate z_{t+1} curve.
- For any z_t, $z_{t+1} \in [z_{min}, z_{max}] \to [z_{min}, z_{max}]$ is the support of the unique and bounded steady state distribution of z.

Implications: target wealth

- For low values of cash at hand, but high enough that the borrowing constraint is not binding, individuals save until they reach the target level z_{max}.
- At high level of wealth the precautionary saving motive is lower and individuals can self-insure nearly perfectly. They dissave above z_{max}.
- At z_{max} the precautionary saving motive is exactly balanced by the consumption tilting motive associated with $\beta(1+r) < 1$.
- Without such counterbalance, the precautionary saving motive would prevail at any level of wealth and the two lines would never cross.

Aggregate asset supply

Obtaining the aggregate asset supply in the limit as $t \to \infty$ requires taking the average across consumers ($\beta = \frac{1}{1+\lambda}$ in the picture)

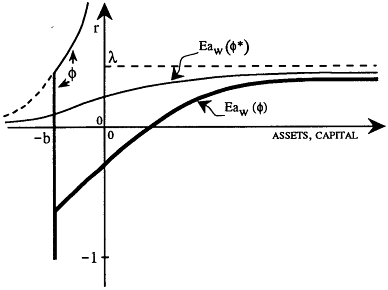

FIGURE IIa
Interest Rate versus Per Capita Assets

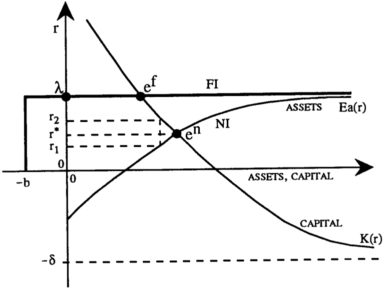

FIGURE IIb
Steady-State Determination

Precautionary saving increases equilibrium asset supply

- Certainty: steady state average asset supply equal $-phi$ if $r < \lambda$ (consumption tilting), is indeterminate if $r = \lambda$ and equals 0 if $r > \lambda$.
- Uncertainty: there is always somebody with wealth above ϕ therefore the aggregate asset supply is alway to the right of $-\phi$.
- As r increases towards the point where $\beta(1 + r) = 1$ $(r = \lambda)$ the limit individual asset supplies diverge to infinity and so must the aggregate one.
- If you add (right hand graph), a downward sloping asset demand, it is clear that the equilibrium interest rate is lower (aggregate asset supply higher) than under certainty. This is due to the precautionary saving motive.