Macroeconomic Policy
Errata and solutions to exercise 6.

1. Errata. In my 1 to 2 class I wrote the government budget identity for an economy with zero seignorage as
$$b_t = \frac{\tau_t - g_t}{1+r} + \frac{b_{t+1}}{1+r}. \tag{1}$$
Imposing solvency this resulted in the government intertemporal budget constraint
$$b_t = \sum_{i=t}^{\infty} \frac{\tau_i - g_i}{(1+r)^{i+1-t}} \tag{2}$$
If we define b_t as the stock of debt at the end of time t, as we did in the lecture notes, then the above two equations are wrong[1]. As in the lecture notes, if b_t is debt at the end of time t the correct version of equation (1) should read
$$b_t = \frac{\tau_{t+1} - g_{t+1}}{1+r} + \frac{b_{t+1}}{1+r}. \tag{3}$$
Correspondingly, the correct version of equation (2) is
$$b_t = \sum_{i=t+1}^{\infty} \frac{\tau_i - g_i}{(1+r)^{i+1-(t+1)}}. \tag{4}$$
These are the same equations, just shifted one period forward and with $\sigma_i = 0$, as equations (6) and (8) in chapter 5 of the lecture notes. In my 2 to 3 class I stuck to the notation in the lecture notes, so you have b_{t-1} rather than b_t.

2. The forward looking form of the government budget identity is
$$b_t = \frac{s_{t+1} + \sigma_{t+1}}{1+r} + \frac{b_{t+1}}{1+r}, \tag{5}$$
with $s_t = \tau_t - g_t$. It can be used to determine the sum of fiscal surplus and seignorage necessary to achieve a given future target level of government debt for given interest rate and initial stock of debt. For example, suppose that the stock of debt at the end of year t is $b_t = 100$b and the government wants to halve it by the end of year $t+1$. This implies $b_{t+1} = 50$b. Knowing the real interest rate r you can use equation (5) to solve for the sum of surplus and seignorage in year $t+1$ necessary to achieve the target. This gives
$$100 = \frac{s_{t+1} + \sigma_{t+1}}{(1+r)} + \frac{50}{1+r} \tag{6}$$
which gives
$$s_{t+1} + \sigma_{t+1} = 100(1+r) - 50. \tag{7}$$

[1] They are correct if, differently from the lecture notes, we define b_t as the stock of debt at the beginning of period t.

The surplus in the current period must equal 50 the reduction in the principal plus $100r$ interests on the outstanding stock of debt.
In the exercise though the government wants to halve the stock of debt not in one year but in five years. So you need to iterate equation (5) up to time $t+5$. This gives

$$b_t = \sum_{i=t+1}^{t+5} \frac{s_i + \sigma_i}{(1+r)^{i+1-(t+1)}} + \frac{b_{t+5}}{(1+r)^5}. \tag{8}$$

Since we assume s_i and σ_i are constant across time we can drop their time index and take them out of the summation. Replacing for the $b_t = 100$ and $b_{t+5} = 50$ gives

$$100 = (s + \sigma) \sum_{i=t+1}^{t+5} \frac{1}{(1+r)^{i+1-(t+1)}} + \frac{50}{(1+r)^5}, \tag{9}$$

which, since r is known, fully determines the level of $s + \sigma$ consistent with achieving the target. Note: work out that the unfriendly summation term in the last equation is nothing more than

$$\sum_{i=t+1}^{t+5} \frac{1}{(1+r)^{i+1-(t+1)}} = \frac{1}{1+r} + \frac{1}{(1+r)^2} + \frac{1}{(1+r)^3} + \frac{1}{(1+r)^4} + \frac{1}{(1+r)^5} \tag{10}$$

3. Do not bother about this question if you are not interest. It deals with stock market bubbles. It just illustrates that the issue of government solvency is related to the issue of whether the price of an asset reflects or not the value of its stream of returns. Equation

$$rp_t = d_t + p_{t+1} - p_t \tag{11}$$

is a no-arbitrage condition. It states that if you are holding a share whose market value at the beginning of period t is p_t it has to be the case that its stream of returns over the current period - dividends d_t plus one-period change in the market price $p_{t+1} - p_t$ - must equal the flow of interests that you could obtain over the same period by selling the share at its market price p_t and investing the proceeds at the interest rate r. Equation can be rewritten as

$$p_t = \frac{d_t}{1+r} + \frac{p_{t+1}}{1+r}, \tag{12}$$

which is similar to the forward-looking version of the government budget identity (be careful that here I have defined p_t as the beginning of period t price of the stock; that is why time indices are slightly different from those on debts and surpluses in the lecture notes). The absence of arbitrage requires the price of the stock to equal the present value of next period dividends plus next period market price. If we substitute in the equation (12) recursively using the equation itself we obtain

$$p_t = \sum_{i=t}^{T} \frac{d_i}{(1+r)^{i+1-t}} + \frac{p_T}{(1+r)^T}. \tag{13}$$

The price of a stock equals the present value of future dividends plus the discounted end-of-time price p_T. If we let $T \to \infty$, equation (13) becomes

$$p_t = \sum_{i=t}^{\infty} \frac{d_i}{(1+r)^{i+1-t}} + \lim_{T\to\infty} \frac{p_T}{(1+r)^T}. \tag{14}$$

If the limit of the second addendum is zero, that is if the price of the stock does not grow at a rate faster than r, the above equation becomes

$$p_t = \sum_{i=t}^{\infty} \frac{d_i}{(1+r)^{i+1-t}}. \tag{15}$$

The price of the stock equals the present discounted value of future dividends (often called fundamentals since the fundamental value of an asset should equal the value of its stream of returns).
If instead the limit of the second addendum is positive, the value of the stock exceeds the present value of its fundamentals. People hold the stock only because they expect its price to grow very fast despite the fact that the stream of payments associated with the stock does not justify the price. In this case the price of the stock is said to contain a bubble. The stock is priced at a value unjustified by its fundamentals only because people expect its price to grow very fast. The bubble is exactly the term

$$\lim_{T\to\infty} \frac{p_T}{(1+r)^T} \tag{16}$$

when the latter is larger than zero. Now, bubbles cannot persist on an easily reproducible asset since agents have an incentive to issue it (are you saying dot.com flotations?) as long as they can sell it at a price in excess of the present value of what they will have to pay back. The increase in the supply of the stock will drive down their market price to the level of fundamentals given by equation (15). In the short run though the bubble may survive until people revise downwards their expectations about the growth in the price of the stock, bursting the bubble and bringing the price down to its fundamental value.