LECTURE 1

1 Preliminaries

1.1 How do macroeconomists think

Macroeconomists (and economists in general) think by means of theories or models → Abstract (mathematical) representations of reality.

Method: Scientific and inductive.

1. Set of facts to explain. Accurately **measure** the variables of interest for such facts.

2. Conjecture a **theory** involving the (endogenous) variables whose behaviour

has to be explained. Two components:

- Assumptions: variables taken as given (exogenous), does not try to explain.
- Logical deductions from the assumptions → implications for endogenous variables.

3. Validate or falsify the theory.

Logical consistency of deductions is necessary but not sufficient for a good theory. A good (useful) theory is not rejected by the data. It is better than an alternative one if:

- fits the data better in a statistical sense;
- can explain a larger set of facts.

A model performs two roles:

- **Measurement tool**: e.g. How big a fraction of y can be explained by x?
- **Laboratory for experiments**: e.g. What happens to y if x changes by 1%?

Components of a model.

- The **model environment**. Exogenous preferences, technology, market structure, property rights.
- The **model equilibrium** . The *vector* of (values for) endogenous variables which satisfies all the conditions of the model.

Example 1: general equilibrium for an endowment economy (static model, just one period). Two agents (A and B) and two goods (1 and 2). Endowments (e_1^A, e_2^A) and (e_1^B, e_2^B).

- Planning equilibrium. Vector (list of numbers) of *allocations* (quantities) of goods supplied and demanded that satisfies the technological constraint (i.e. for each good total consumption does not exceed total endowment). In symbols, a vector $\{e_1^A, e_2^A, e_1^B, e_2^b, c_1^A, c_2^A, c_1^B, c_2^B\}$ such that $c_1^A + c_1^B \leq e_1^A + e_1^B$ and $c_2^A + c_2^B \leq e_2^A + e_2^B$.
- Market equilibrium. Vector of allocations *and prices* such that consumers maximize their utility and all markets clear (demand equal supply). In symbols, a vector $\{e_1^A, e_2^A, e_1^B, e_2^B, c_1^A, c_2^A, c_1^B, c_2^B, p_1, p_2\}$ such that consumers maximize utility and $c_1^A + c_1^B = e_1^A + e_1^B$ and $c_2^A + c_2^B = e_2^A + e_2^B$.

Example 2: general equilibrium for the same endowment economy (dynamic model, many periods indexed by t). Endowments $(e_1^A(t), e_2^A(t))$ and $(e_1^B(t), e_2^B(t))$.

The model equilibrium is no longer a *vector* (list of number) for the endogenous variables. It is a *vector of functions of time;* i.e. one list of numbers for the endogenous variables for *each time period*(indexed by time).

- Planning equilibrium. $\{e_1^A(t), e_2^A(t), e_1^B(t), e_2^B(t), c_1^A(t), c_2^A(t), c_1^B(t), c_2^B(t)\}$ such that $c_1^A(t) + c_1^B(t) \leq e_1^A(t) + e_1^B(t)$ and $c_2^A(t) + c_2^B(t) \leq e_2^A(t) + e_2^B(t)$.

- Market equilibrium: $\{e_1^A(t), e_2^A(t), e_1^B(t), e_2^B(t), c_1^A(t), c_2^A(t), c_1^B(t), c_2^B(t), p_1(t), p_2(t)\}$ such that consumers maximize utility and $c_1^A(t) + c_1^B(t) = e_1^A(t) + e_1^B(t)$ and $c_2^A(t) + c_2^B(t) = e_2^A(t) + e_2^B(t)$.

Model macroeconomics is inherently dynamics because:

- Present affects the future through accumulation.
- If agents are forward looking, future affects the present through expectations.

$\longrightarrow$ Nearly all the equilibria we will study in macroeconomics will be *vectors of functions of time.*

2 A brief (and easy) introduction to differential equations

- The solution to an **algebraic equation** (or system of them) is a *number* or a vector of *numbers*.

- The solution to a **differential equation** (or system of them) is a *function* or vector of *functions*.

Macroeconomics deals with dynamic models whose equilibria (solutions) are vectors *functions of time* $\rightarrow$ the equations characterizing macro models are <u>differential equations</u> (difference equations if time is discrete).

2.1 A few examples and the general problem

Let $\dot{x}(t) = \frac{dx(t)}{dt}$ denote the first derivative of the function $x(t)$ with respect to its argument t.

Example 1: The simplest differential equation is

$$\dot{x}(t) = 0. \tag{1}$$

1. A differential equations contains the derivative of a function.
2. Solution: a function $x(t)$ which together with its derivative satisfies the equation.
3. The solution $x(t)$ is a *smooth* function of t since it has a derivative.

- General solution: $x(t) = a$ for any a. The general solution describes *all* functions which satisfy (1). They are an infinite number (any horizontal line). But there is just one line with a given slope which goes through a particular point.

- Particular solution: a *unique* function $x(t)$ satisfying (1) (i.e. belonging to the family $x(t) = a$) *and* passing through some point $(\hat{t}, x(\hat{t}))$.

  E.g. if $(\hat{t}, x(\hat{t})) = (1, 4)$. The particular solution is $x(t) = 4$.

To go from the general to the particular solution we need one (or more) **boundary conditions** (i.e. points on the function).

Example 2: Find the general and particular solution to the differential equation

$$\dot{x}(t) = 3x(t) \tag{2}$$

with boundary condition (1,1).

A differential equation is of order n if it contains the nth derivative of the unknown function.

The differential equations (1) and (2) were first-order.

Example 3: Consider now the second order difference equation

$$\ddot{x}(t) = 3 \tag{3}$$

It is straightforward to verify that the general solution has the form

$$x(t) = a + bt + 3^2 \tag{4}$$

for any a and b.

There are now two unknown parameters and we need **two** boundary conditions to solve for the particular solution.

Remark: One needs as many boundary conditions as the order of the differential equation to solve for its particular (unique) solution.

2.2 Solving differential equations

Solving differential equations is a bit of an art!

We will use two main methods in this course:

1. Guess and verify. This is what we did in the previous three examples. We guess a function, substitute in the equation and verify that it is a solution.
2. Graphical solution. Phase diagram: geometrical representation of direction of movement as a function of value of x. Two components:
   - Stationary state: value of x which if reached at any time t_0 implies $x(t)$ is constant for any $t \geq t_0$.
   - Dynamics away from the steady state.

3 Useful mathematical results

3.0.1 Derivative of a function of a function

Be a function $y = h(x) = g(f(x))$. That is $h(x)$ is the composition of the function $g(z)$ and $z = f(x)$ [e.g. $y = log(x^2)$ is the composition of $log(z)$ and $z = x^2$]. Then dy/dx, the derivative of y with respect to x, equals the derivative of the first function time the derivative of the second function. More formally

$$\frac{dy}{dx} = \frac{dg}{dz} \cdot \frac{dz}{dx}. \quad (5)$$

For example, for $y = log(x^2)$ it is $dlog/dz = 1/z = 1/x^2$, $dx^2/dx = 2x$ and

$$\frac{dy}{dx} = \frac{2x}{x^2}. \quad (6)$$

3.0.2 Derivative of implicit functions

Suppose you *know*[1] that a variable y is a differentiable function $y = f(x)$ of some variable x but that the functional relationship between the two variables is available only in the implicit form $g(x, y) = 0$ (e.g. $x^3 + y^3 - 3 = 0$).

The derivative $dy/dx = f'(x)$ of the unknown function $y = f(x)$ with respect to x can be calculated as

$$\frac{dy}{dx} = -\frac{\partial g(x, y)/\partial x}{\partial g(x, y)/\partial y}. \tag{7}$$

To see this note that $g(x, y) = g(x, f(x)) = 0$ for any x. Since $g(x, f(x)) = 0$ for any x the derivative of the two sides of the equation must be the same for any x. The derivative of zero is zero and so the derivative of the LHS must also

[1]Note that we are assuming that y is a function of x. In general the existence of a relationship $g(x, y) = 0$ does not imply that y is a function of x (e.g. $x^2 + y^2 = 1$ is not a function). The implicit function theorem provides a sufficient condition for this to be the case and such conditions while always be verified in the cases we will consider in the course.

be zero. Using the rule for differentiating a function of a function and equating to zero we have

$$\frac{dg(x, f(x))}{dx} = \frac{\partial g(x, y)}{\partial x} + \frac{\partial g(x, y)}{\partial y}\frac{dy}{dx} = 0. \tag{8}$$

which can be rearranged to obtain (7).

3.0.3 Properties of logarithms

- Log of a product.

$$\log(xy) = \log(x) + \log(y). \tag{9}$$

- Log of a power.

$$\log(x^{\alpha}) = \alpha \log(x). \tag{10}$$

- Log of a quotient [2].

$$\log(x/y) = \log(x) - \log(y). \tag{11}$$

3.0.4 Logs and rates of growth

Suppose a variable $X(t)$ is a function of time. Then the derivative of $\log X(t)$ with respect to time equals the percentage change in the variable over time. To see this apply the rule for the derivative of a function of a function to $y = \log X(t)$ to get

$$\frac{d \log X(t)}{dt} = \frac{1}{X(t)} \frac{dX(t)}{dt}. \tag{12}$$

$dX(t)/dt$ is the absolute value of the change in $X(t)$ when time changes by a small amount dt. $X(t)^{-1} dX(t)/dt$ is the change divided its initial value; i.e. the

[2] This follows from the previous two by noticing that $x/y = xy^{-1}$.

percentage change. In what follow we will use the notation $\dot{X}(t)$ for $dX(t)/dt$. We can now derive the rates of growth for various functions of variable which depend on time.

- Rate of growth of product

$$\frac{(X(t)\dot{Y}(t))}{X(t)Y(t)} = \frac{\dot{X}(t)}{X(t)} + \frac{\dot{Y}(t)}{Y(t)} \tag{13}$$

- Rate of growth of power.

$$\frac{(X(\dot{t})^{\alpha})}{X(t)} = \alpha\frac{\dot{X}(t)}{X(t)} \tag{14}$$

- Rate of growth of quotient

$$\frac{(X(t)\dot{/}Y(t))}{X(t)/Y(t)} = \frac{\dot{X}(t)}{X(t)} - \frac{\dot{Y}(t)}{Y(t)} \tag{15}$$

4 Structure of the course

- First part: "full employment economy". No money (only real variables). Same results if money but no nominal rigidities (money neutrality).
- Second part: deviations from full employment and monetary non-neutrality. This requires some nominal (measured in unit of money) variable to be rigid.

5 Road map for the growth literature

5.1 Motivation/stylized facts

- Standard of living (income level per head) have increased dramatically over the past two hundred years in most developed countries. Income per capita was hardly above subsistence until the industrial revolution. Growth rates have also been increasing over the past two centuries (growth was very low by today's standard before the industrial revolution)
- Large differences in income levels and growth rates over space too.
  - Understanding these large differences in *income levels* is understanding the large differences in standard of living we observe. E.g. In 1992 according to the Penn Data tables, per capita income in the US was 45 times that of Chad (poorest country in the world).

Crucial to understand whether it is possible to intervene to raise standards of living.

– Understanding differences in *growth rates* is possibly *the* most important question in macroeconomics, as the effect of theses differences compounds over time (e.g. if the UK had grown at 1 rather that 2% a year in the last century income per capita would be roughly half of what it is now).

- What happens to the world income distribution over time?

Per capita income in national economies	times world per capita income 1960–64	1985–89
10th %-ile	0.22 ×	0.15 ×
	(26.0% world popn.)	(3.3% world popn.)
90th %-ile	2.70 ×	3.08 ×
	(12.5% world popn.)	(9.3% world popn.)
(25th-15th) %-iles	0.13 ×	0.06 ×
(95th-85th) %-iles	0.98 ×	0.59 ×

Table 4: Cross-country distribution dynamics in per capita incomes

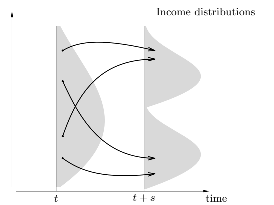

Fig.: Countries in the world, emerging twin peaks. Arrows show country per capita incomes transiting across different parts of the cross-section distribution

The fortunes of individual countries sometimes change (growth miracles and growth disasters). Unclear whether the distribution is rather stable or it is changing with a tendency to become bimodal (twin peaked). The spread of the distribution clearly increased over the past 200 years, yet stable over the last 30.

- Growth more volatile in poor than rich countries.
- Various factors correlated with growth: investment rates, education, political instability, enforcement of property rights.

5.2 Questions

- What explains large differences in income *levels* ?
- What explains large differences in income *growth rates* ?
- Why are differences persistent?
- What drives overall world growth?

5.3 Some answers

- Per capita income differences must be due to differences in the quantities of inputs used or differences in technologies (somewhat restrictively defined).
- *Growth accounting* decomposes income changes across time between changes in input and changes in technology. *Development accounting* does the same for income level differences across countries. Just an *accounting* decomposition, though. They do not take into account the endogeneity of inputs [3].

[3] Some technical problems to solve (won't talk much about it): how to measure inputs (perpetual inventory method for capital, men hours for labour, adjusting for input quality).

- Solow model (neoclassical growth theory): can capital accumulation *explain* observed differences in:
  1. growth rates?
  2. income levels?

  If capital does not affect technological efficiency, the answer is a definite no for question 1 and no, provided capital is paid its marginal product(or close to it), for question 2. $\rightarrow$ Differences must then be due to differences in technologies.

- Endogenous growth: what explains differences in technological efficiency and differences in growth rates? Unlikely to be able to explain cross-country differences if technology can be transferred across countries.$\longrightarrow$ Hence, barriers/disincentives to technological adoption must be an important part of the explanation.

6 Accounting for growth and income differences

Consider an economy using three inputs (labour L, capital K and "efficiency" A) to produce one output good Y. This can be described by the production function $Y = F(K, AL)$.

Suppose the function F is the same across countries. Then if different countries have different income levels they must use different quantities of one of the three inputs. A similar reasoning applies to rates of growth. We can get a bit more insight by adding a bit more structure.

6.1 Accounting for growth differences.

Taking logs and time derivatives of the production function we can write

$$\frac{\dot{Y}}{Y} = \frac{F_K}{Y}\dot{K} + \frac{F_L}{Y}\left(A\dot{L} + L\dot{A}\right) = \frac{F_K K}{Y}\frac{\dot{K}}{K} + \frac{AF_L L}{Y}\left(\frac{\dot{L}}{L} + \frac{\dot{A}}{A}\right), \tag{16}$$

where F_i denotes the partial derivative of F with respect to the variable i.

If inputs receive their marginal product then $\frac{F_K K}{Y}$ and $\frac{F_L L}{Y}$ are respectively the share of capital and labour income in total income. If the production function has constant returns then the two shares add up to one and if we denote the capital share at time t by $\alpha(t)$ we can rewrite (16) as

$$\frac{\dot{Y}}{Y} = \alpha(t)\frac{\dot{K}}{K} + (1 - \alpha(t))\left(\frac{\dot{L}}{L} + \frac{\dot{A}}{A}\right), \tag{17}$$

which can be rearranged as

$$\frac{\dot{Y}}{Y} - \frac{\dot{L}}{L} = \alpha(t)\left(\frac{\dot{K}}{K} - \frac{\dot{L}}{L}\right) + (1 - \alpha(t))\frac{\dot{A}}{A}. \tag{18}$$

Note that by the properties of growth rates the LHS is the rate of growth of per capita income Y/L. Equation (18) implies that if two countries with the same production function F have different growth rates then they must have different rates of growth either for capital per worker K/L or the factor A. A is often called technological progress, TFP (total factor productivity) or efficiency. Yet, since the equation is an accounting identity must hold at all times it is in fact a residual (anything else), the term necessary to reconcile the observed growth in measured income with that in measured capital and labour. For this reason A goes also under the name of Solow residual.

The accounting identity (18) makes clear that to explain differences in per capita income growth across time or countries we have to look at two possible candidates:

- differences in the rate of growth of capital per head
- differences in the the rate of growth of efficiency (be it due to differences in knowledge, education).

An accounting identity is not an explanation. It cannot tell us for example what determines growth in capital per head and in total factor productivity, neither whether one causes the other. So we need theories to explain differences in per capita income growth.

- The **neoclassical growth theory** (started by Solow):
  - A as exogenously given
  - to what extent growth in capital per head can explain differences in income growth rates.
- The new **endogenous growth theory** endogenizes A (i.e. it tries to explain per capita income growth differences in terms of differences in A and its rate of growth).

6.2 Accounting for income level differences

This requires some additional structure. We specialize the production function to a Cobb-Douglas; i.e.

$$Y = K^{\alpha}(AL)^{1-\alpha}. \tag{19}$$

By dividing both sides by Y income per worker can be written as

$$\frac{Y}{L} = \left(\frac{K}{L}\right)^{\alpha} A^{1-\alpha}. \tag{20}$$

If countries have the same efficiency A to explain a ratio of income per worker of x requires a ratio of capital per worker $x^{1/\alpha}$, where $\alpha < 1$. If $x = 10$ and $\alpha = 0.3$ the associated ratio of capital per worker is 1000!

Alternatively, one can divide both sides of (19) by K to obtain

$$\frac{Y}{K} = \left(\frac{K}{AL}\right)^{\alpha-1} \tag{21}$$

Solving for K/L and replacing in (20) gives

$$\frac{Y}{L} = \left(\frac{Y}{K}\right)^{\frac{\alpha}{\alpha-1}} A \tag{22}$$

Note that Y/K is the average product of capital, with a Cobb-Douglas production function the marginal product of capital equal α times the average product. If countries have the same A to explain a ratio of income per worker of 10 we need a ratio of roughly 100 in rates of return.

7 The Solow growth model

7.1 Kaldor stylized facts

Kaldor (1963) first provided some stylized facts concerning the relationship between capital accumulation and growth.

1. The stock of physical capital per worker grows over time.
2. The ratio between output and the stock of capital is constant in the long run.
3. The shares of capital and labour in national income are nearly constant.
4. The growth rate of output per worker differs substantially across countries.

The Solow growth model can claim a good record in fitting facts 1-3 and some success in fitting 4.

7.2 The environment

- Continuous time
- Unique consumption good that can only be turned one-to-one into capital

input (and back). No adjustments (installation/deinstallation) costs.

- Three inputs: capital K, labour L and TFP A.

- Capital depreciates at exogenous rate $\delta > 0$.

- Infinitely lived working age population growing at rate growing at rate $n \longrightarrow$

$$L_t = L_0 e^{nt}. \tag{23}$$

- TFP reproduces (grows) at the exogenous rate $g \longrightarrow$

$$A_t = A_0 e^{gt}. \tag{24}$$

- Production function $Y = F(K, AL)$ with the following properties

1. Increasing in each input but decreasing marginal returns

$$\frac{\partial F}{\partial K}, \frac{\partial F}{\partial L} > 0 \tag{25}$$

$$\frac{\partial^2 F}{\partial K^2}, \frac{\partial^2 F}{\partial L^2} < 0 \tag{26}$$

2. Constant returns to scale in K and labour measured in efficiency units AL

$$F(\lambda K, \lambda AL) = \lambda F(K, AL) \tag{27}$$

Let $Z/AL = \tilde{z}$. CRS implies

$$F(K, AL) = ALF(K/AL, 1) \tag{28}$$

$$\tilde{y} = f(k), \qquad f'(k) >, \qquad f''(k) < 0. \tag{29}$$

It is straightforward to check that

$$F_K = f'(k) \tag{30}$$

$$F_L = f(k) - kf'(k). \tag{31}$$

3. Inada conditions

$$\lim_{K\to 0} F_K = \lim_{L\to 0} F_L = \infty \tag{32}$$

$$\lim_{K\to \infty} F_K = \lim_{L\to \infty} F_L = 0 \tag{33}$$

- Identical agents. No trade and the aggregate economy is just the sum of each Robinson Crusoe agent producing, consuming and saving its own output.

- Exogenous saving rate s

- Fundamental capital accumulation equation

$$\dot{K} = sF(K, AL) - \delta K = sALf(k) - \delta K. \tag{34}$$

7.3 Steady equilibrium (balanced growth path)

Definition 1 *An equilibrium is a sequence of (allocation) vectors* $\{L_t, K_t, A_t, Y_t\}$ *such that* (23),(24),(34) *and the production function are satisfied.*

Definition 2 *A steady state equilibrium (or balanced growth path) is a sequence of (allocation) vectors* $\{L_t, K_t, A_t, Y_t\}$ *such that* (23),(24),(34) *and the production function are satisfied* and *all variables grow at a constant rate.*

To find the stationary equilibrium divide both sides of (34) by K to obtain

$$\frac{\dot{K}}{K} = s\frac{f(k)}{k} - \delta. \tag{35}$$

From this, using the properties of growth rates we have $\dot{k}/\tilde{k} = \dot{K}/K - n - g$ and

$$\frac{\dot{k}}{k} = s\frac{f(k)}{k} - \delta - n - g. \tag{36}$$

The LHS must be constant on the balanced growth path, but the RHS is strictly decreasing in k. So if a balanced growth path exists it has to feature constant value of k or $\dot{k}/k = 0$.

Therefore on the balanced growth path capital per worker K/L grows at the same rate g as technological progress.

The steady state equilibrium value of k is k^* satisfying

$$\frac{f(k^*)}{k^*} = \frac{\delta + n + g}{s}. \tag{37}$$

This implies that the output/capital ratio is constant and that also output per worker grows at the exogenous rate of technological progress g. Growth in income per worker is associated with growth in capital per worker. Yet, the *cause* of both is technological progress. Without technological progress both variables are constant in steady state.

If economies are never far away from the steady state equilibrium, the Solow growth model cannot *explain* why different economies grow persistently at different rates. For this to be the case their stocks of technological progress must grow at different rates.

7.4 Non-stationary equilibrium (transitional dynamics)

To solve for the off-steady-state equilibrium consider again equation (36). The first term on the RHS is declining in k the second term is a constant. So for $k < k^*$ the difference between the two quantities is positive and declining and $\dot{k}/k > 0$ until k^* is reached. Vice versa for $k > k^*$.

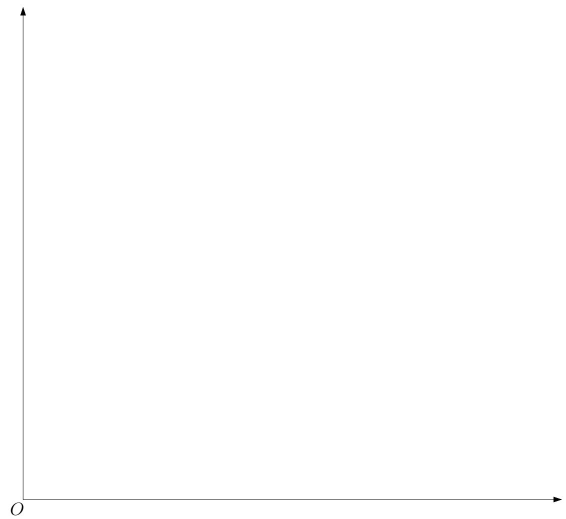

Figure 1: Dynamics of k : gross investment rate versus effective depreciation rate

The picture conveys an important intuition. A unit of k effectively depreciates at rate $\delta{+}n{+}g$[4]. The investment capital ratio $sf(k)/k$ is decreasing in k. When k is relatively low it is high, and its exceeds the depreciation rate. But because capital (the reproducible factor) is subject to decreasing returns, eventually the investment rate just equals the depreciation rate.

[4]Given an initial value of K if $s = 0$, k falls at $\delta + n + g$

The same point can be made in terms of investment and depreciation flows rather than rate by multiplying both sides of (36) by k to obtain

$$\dot{k} = sf(k) - (\delta - n - g)k. \tag{38}$$

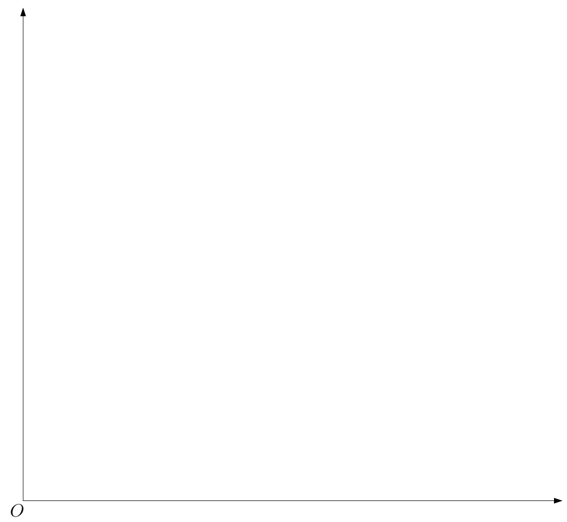

Figure 2: Dynamics of k : gross investment vs replacement investment