Macroeconomics A

Solution to problem set 5

1. The solvency constraint (20) in the lecture notes is

$$\lim_{t\to\infty} b_t e^{-R_t+(n+g)t} \geq 0. \tag{1}$$

Market clearing implies $b_t = k_t$ and firm's optimization requires $r_t = f'(k_t) + \delta$. In steady state r_t is constant and (1) can be written as

$$\lim_{t\to\infty} k^* e^{[-f'(k^*)-\delta+(n+g)]t} = \lim_{t\to\infty} k^* e^{[-\rho-\theta g+(n+g)]t} \geq 0, \tag{2}$$

where the first equality comes from evaluating the Euler equation in steady state. We know that $-\rho - \theta g + (n+g) < 0$, since it is the condition for boundedness of the household's lifetime utility. Hence, the limit (2) is zero.

2. If we denote by small letters variables measured in efficiency units of labour the production function in intensive form is $y_t = k_t^\alpha$.

   (a) In intensive units the consumer problem is

$$U_0 = \max_{c_t} \int_0^\infty \frac{c_t^{1-\theta}}{1-\theta} e^{-[\rho-(1-\theta)g-n]t} dt \tag{3}$$

$$\text{s.t. } \dot{b}_t = (r_t - n - g)b_t + w_t - c_t \tag{4}$$

$$\lim_{t\to\infty} b_t e^{-R_t+(n+g)t} \geq 0 \tag{5}$$

$$b_0 \text{ given.} \tag{6}$$

   Equation (4) and (5) can be used to derive the intertemporal budget constraint.

$$\int_0^\infty c_t e^{-R_t+(n+g)t} dt \leq b_0 + \int_0^\infty w_t e^{-R_t+(n+g)t} dt. \tag{7}$$

   Therefore the consumer problem can also be written as maximizing equation (3) subject to the intertemporal budget constraint. The associated Lagrangean is

$$\mathcal{L} = \int_0^\infty \frac{c_t^{1-\theta}}{1-\theta} e^{-[\rho-(1-\theta)g-n]t} dt + \lambda \left[b_0 + \int_0^\infty (w_t - c_t) e^{-R_t+(n+g)t} dt \right]. \tag{8}$$

   The sequence of FOCs, one for each t, is

$$c^{-\theta} e^{-[\rho-(1-\theta)g-n]t} = \lambda e^{-R_t+g+nt}, \tag{9}$$

which simplifies to

$$c^{-\theta} = \lambda e^{-[R_t-(\rho+\theta g)t]}. \tag{10}$$

Taking logs and time derivatives we obtain the Euler equation

$$\frac{\dot{c}}{c} = \frac{r_t - \rho - \theta g}{\theta}. \tag{11}$$

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(b) Factor market equilibrium implies $b_t = k_t$, $f'(k_t) = r_t + \delta$ and $w_t = f(k_t) - f'(k_t)k_t$. Replacing in (11) the latter implies

$$\frac{\dot{c}}{c} = \frac{f'(k_t) - \rho - \delta - \theta g}{\theta}. \tag{12}$$

. While replacing in the dynamics constraint we obtain

$$\dot{k}_t = f(k_t) - c_t - (\delta + n + g)k_t. \tag{13}$$

(c) The curve shifts down.

(d) The curve shifts left. For $\theta \to 0$ the shift is negligibly small.

(e) See graph in class.

(f) Using (13) evaluated in steady state we have:

$$s^* = 1 - \left(1 - (\delta + n + g)\frac{k^*}{f(k^*)}\right) = (\delta + n + g)\frac{\alpha}{f'(k^*)} \tag{14}$$

$$= \alpha\frac{\delta + n + g}{\rho + \delta + \theta g}. \tag{15}$$

A higher g increases s^* if

$$\frac{\partial s^*}{\partial g} = \alpha\frac{\rho + \delta + \theta g - \rho - n - g}{(\rho + \delta + \theta g)} > 0 \tag{16}$$

or

$$\delta - n - (1 - \theta)g > 0. \tag{17}$$