ECOM 009 Macroeconomics B

Lecture 7

Giulio Fella

Plan for the rest of this lecture

- Introducing the general asset pricing equation
- Consumption-based asset pricing.
- Consumption-based asset pricing in general equilibrium:
  - Lucas's tree model
  - Some applications
- Readings: Ljungqvist and Sargent, Ch. 13.1-13.3, 13.5-13.8

The basic pricing equation

Consider the following equation

$$p_t^j(z_t) = \mathbb{E}_t[m_{t+1}x_{t+1}^j] = \sum_{i=1}^{n} \pi(z_{t+1}|z_t)m(z_{t+1}|z_t)x^j(z_{t+1}) \quad (142)$$

- $z_t \in \{z_1, z_2, \ldots, z_n\}$ is the value of the state variable at time t. We assume it is first-order Markov.
- $p_t^j(z_t)$ is the price of asset j at time t in state z_t
- x_{t+1}^j is the payoff of asset j at time $t+1$
- $m_{t+1} = m(z_{t+1}|z_t)$ is the price (in units of the numeraire) in the current state z_t and time t of one (certain) unit of numeraire in state z_{t+1} at time $t+1$.

The basic pricing equation II

- The price of an asset is the expected value of the future state-contingent asset payoffs times the state-contingent prices.
- The state-contingent pricing function $m(z_{t+1}|z_t)$ is called the *stochastic discount factor* or *pricing-kernel.*
  - The stochastic discount factor is the same for all assets!
  - Relative price of the numeraire across time and states.
- Different asset pricing theories involve different assumptions on what determines the stochastic discount factor.

The basic pricing equation: an example

Consider a deterministic economy.

- $z_t = z_1$ for all t.
- The basic pricing equation becomes

$$p_t^j = m(z_1)x_{t+1}^j$$

- p_t^j is the discounted present value of the future payoff x_{t+1}^j
- $m(z_1)$ is the price of one unit of numeraire tomorrow in terms of numeraire today (the market discount rate)
- $m(\cdot)$ is called the (stochastic) discount factor because it generalizes the above notion to a stochastic environment.

Breaking down the basic pricing equation

One can rewrite the pricing equation as

$$p_t^j(z_t) = \mathbb{E}_t[m_{t+1}x_{t+1}^j] = \mathbb{E}_t(m_{t+1})\mathbb{E}_t x_{t+1}^j + cov\left(m_{t+1}, x_{t+1}^j\right)$$

- If x_{t+1}^j is uncorrelated with m_{t+1} (e.g. x_{t+1}^j is deterministic; i.e. risk-free) the covariance term is zero.
  - Same as in the deterministic case with the only difference that $m(z_1)$ is replaced by $\mathbb{E}_t(m_{t+1})$.
    (Note: z_{t+1} and, consequently, m_{t+1} are NOT deterministic).
- $p_t^j(z_t)$ is higher (lower) if the covariance term is +ve (-ve).
  - An asset for which x_{t+1}^j is high in states with high price m_{t+1} has a higher price today.

Consumption-based asset pricing

Effectively, **the** macroeconomic theory of asset pricing.

With two assets the optimization problem for consumer k is

$$W(L_{t-1}^k, N_{t-1}^k, z_t) = \max_{c_t^k, L_t^k, N_t^k} u(c_t^k) + \beta E W(L_t^k, N_t^k, z_{t+1}) \quad (143)$$

$$\text{s.t. } c_t^k + R_t^{-1} L_t^k + p_t N_t^k = L_{t-1}^k + (p_t + y_t) N_{t-1}^k \quad (144)$$

$$L_t^k, N_t^k \geq 0, \; L_{t-1}^k, N_{t-1}^k \text{ given} \quad (145)$$

- L_t^k and N_t^k choice of stock of risk-free and risky asset.
- Future return on risky asset $(y_{t+1} + p_{t+1})/p_t$ is stochastic.
- No labour income (for simplicity)

Euler equations

Replacing for c_t^k, maximizing with respect to L_t^k and N_t^k and using the envelope condition, yields the Euler equations

$$R_t^{-1} = E_t \left[\beta \frac{u'(c_{t+1}^k)}{u'(c_t^k)} \right] \tag{146}$$

$$p_t = E_t \left[\beta \frac{u'(c_{t+1}^k)}{u'(c_t^k)} (p_{t+1} + y_{t+1}) \right] \tag{147}$$

- Those Euler equations are basic asset pricing equations with $m_{t+1}^k = \beta \frac{u'(c_{t+1}^k)}{u'(c_t^k)}]$
- In consumption-based asset pricing theories the stochastic discount factor is the MRS between consumption today and tomorrow.
- Unique prices (no arbitrage) only if $m_{t+1}^k = m_{t+1}$ for all k.

Risky assets

$$p_t = E_t[m_{t+1}(p_{t+1} + y_{t+1})] = E_t\left[\beta \frac{u'(c^k_{t+1})}{u'(c^k_t)}(p_{t+1} + y_{t+1})\right]$$

- The price of a risky asset is higher (its expected return lower) the more it pays in states in which m_{t+1} is high; i.e. in which c_{t+1} is low.
- Such an asset provides more insurance in worse state.
- Desirable, hence $p_t \uparrow$

Only non-diversifiable risk matters

The pricing equation can also be written as

$$\begin{aligned} p_t =& E_t[m_{t+1}(p_{t+1} + y_{t+1})] \\ =& E_t[m_{t+1}]E_t[(p_{t+1} + y_{t+1})] + cov[m_{t+1}, (p_{t+1} + y_{t+1})] \end{aligned}$$

- Idiosyncratic risk (covariance term is zero) does not affect prices
- No need for compensation as risk is fully diversified.
- Only risk correlated with future consumption affects prices (and returns).

When are stock prices a martingale?

$$
\begin{aligned}
p_t =& E_t[m_{t+1}(p_{t+1} + y_{t+1})] \\
=& E_t[m_{t+1}]E_t[(p_{t+1} + y_{t+1})] + cov[m_{t+1}, (p_{t+1} + y_{t+1})]
\end{aligned}
$$

- Two necessary conditions for p_t to be a martingale
  - $E_t m_{t+1} = E_t[\beta u'(c_{t+1})/u'(c_t)]$ is a constant
  - The covariance term is zero
- If agents are risk-neutral $E_t m_{t+1} = \beta$ and

$$
p_t = E_t \beta(p_{t+1} + y_{t+1}) = \sum_{j=1}^{\infty} \beta^j E_t y_{t+j} + \lim_{k \to \infty} \beta^k E_t p_{t+k}
$$

- The last (bubble) term equals zero in the general equilibrium models we consider.

Consumption-based asset pricing in general equilibrium

- Our pricing equations are informative on prices only to the extent that we know the stochastic discount factor.
- To determine the SDF all models proceed along the following lines.
  1. Postulate an economic environment.
  2. Derive the equilibrium allocation.
  3. Assume competitive markets for the assets of interests and solve for the agents' Euler equations $\rightarrow$ pricing equations.
  4. Impose that the consumption allocation in 2. coincides with consumers' demand in 3.

Lucas's tree model

- One good: coconuts (non-storable).
- Two assets: equity (ownership of one coconut trees) and bonds.
- All coconut trees yield the same payoff $y_t(z_t)$ in state z_t (aggregate shocks).
- All agent are identical: same preferences and initial endowment of one tree
- Trivial equilibrium: each agent consumes the current flow of coconuts from her tree and net zero bond supply.

Equilibrium asset prices in Lucas model

Same pricing equations but now consumption is determined.

$$R_t^{-1} = E_t\left[\beta\frac{u'(y_{t+1})}{u'(y_t)}\right]$$

$$p_t = E_t\left[\beta\frac{u'(y_{t+1})}{u'(y_t)}(p_{t+1} + y_{t+1})\right]$$

Stock prices in Lucas model

One can iterate on the second equation to obtain

$$u'(y_t)p_t = E_t \sum_{j=1}^{\infty} \beta^j u'(y_{t+j}) y_{t+j} + E_t \lim_{k \to \infty} \beta^k u'(y_{t+k}) p_{t+k}$$

- For agents to be willing to hold their tree forever in equilibrium the last term has to be zero (no bubble).
  - Suppose not...
- The expression for the equilibrium stock price can be written as

$$p_t = E_t \sum_{j=1}^{\infty} \beta^j \frac{u'(y_{t+j})}{u'(y_t)} y_{t+j}$$

A special case

Suppose that $u(c) = log(c)$.

- The expression for the stock price becomes

$$p_t = E_t \sum_{j=1}^{\infty} \beta^j \frac{y_t}{y_{t+1}} y_{t+j} = \frac{\beta}{1-\beta} y_t$$

- Remark: different choices of functional forms for u imply different asset pricing models.

The term structure of interest rates

Let's now introduce a second (two-period) risk-free bond

$$W(L_{1,t-1}, L_{2,t-1}N_{t-1}, y_t) = \max_{c_t, L_{1,t}, L_{2,t}N_t} u(c_t) + \beta E W(L_{1,t}, L_{2,t}, N_t, y_{t+1})$$

$$\text{s.t. } c_t + R_{1,t}^{-1} L_{1,t} + R_{2,t}^{-1} L_{2,t} + p_t N_t = L_{1,t-1} + R_{1,t}^{-1} L_{2,t-1} + (p_t + y_t) N_{t-1}$$

$$L_{1,t}, L_{2,t}, N_t \geq 0, \; L_{1,t-1}, L_{2,t-1}, N_{t-1} \text{ given}$$

- $R_{1,t}^{-1}$ and $R_{2,t}^{-1}$ are the current prices of a bond with respectively a one-period and two-period remaining maturity.
- Absence of arbitrage requires the time-t price of a two-period bond issued last period to be $R_{1,t}^{-1}$
- It follows that the one-period-ahead return of a newly-issued two-period bond is uncertain.

Bond pricing equations

Imposing equilibrium ($c_t = y_t$), the Euler (or pricing) equations for the two bonds can be written as

$$R_{1,t}^{-1} = \beta E_t \left[\frac{u'(y_{t+1})}{u'(y_t)} \right]$$
$$R_{2,t}^{-1} = \beta E_t \left[\frac{u'(y_{t+1})}{u'(y_t)} R_{1,t+1}^{-1} \right] = \beta^2 E_t \left[\frac{u'(y_{t+2})}{u'(y_t)} \right]$$

- The first equality in the second equation can be written as

$$R_{2,t}^{-1} = \beta E_t \left[\frac{u'(y_{t+1})}{u'(y_t)} \right] E_t R_{1,t+1}^{-1} + cov \left[\beta \frac{u'(y_{t+1})}{u'(y_t)}, R_{1,t+1}^{-1} \right]$$
$$= R_{1,t}^{-1} E_t R_{1,t+1}^{-1} + cov \left[\beta \frac{u'(y_{t+1})}{u'(y_t)}, R_{1,t+1}^{-1} \right]$$

The pure expectation theory of the term structure

$$R_{2,t}^{-1} = R_{1,t}^{-1} E_t R_{1,t+1}^{-1} + cov\left[\beta \frac{u'(y_{t+1})}{u'(y_t)}, R_{1,t+1}^{-1}\right]$$

- The first addendum embodies the pure expectation theory of the term structure.
    - Long rates are just a (geometric) average of expected future short rate.
    - $R_{2,t} > R_{1,t}$ if rates are expected to increase.
- The pure expectation theory holds exactly *only if* the covariance term is zero. E.g.
    - Risk-neutral agents
    - No uncertainty

A different look at the term structure

The pricing equation

$$R_{2,t}^{-1} = \beta^2 \left[\frac{E_t u'(y_{t+2})}{u'(y_t)} \right]$$

generalizes to a $j-$period bond

$$R_{j,t}^{-1} = \beta^j \left[\frac{E_t u'(y_{t+j})}{u'(y_t)} \right]$$

It can be written in terms of returns rather than prices as

$$R_{j,t} = \beta^{-j} \left[u'(y_t)[E_t u'(y_{t+2})]^{-1} \right]$$

A different look at the term structure II

The corresponding annual implied return is

$$\tilde{R}_{j,t} = R_{j,t}^{1/j} = \beta^{-1}\left[u'(y_t)[E_t u'(y_{t+j})]^{-1}\right]^{1/j}$$

- If dividends are i.i.d. the expectation term $E_t u'(y_{t+j}) = Eu'(y_t)$ is constant for all $j > 0$ and we can write

$$\frac{\tilde{R}_{j,t}}{\tilde{R}_{k,t}} = \left[u'(y_t)[Eu'(y)]^{-1}\right]^{\frac{1}{j}-\frac{1}{k}} = \left[u'(y_t)[Eu'(y)]^{-1}\right]^{\frac{k-j}{kj}}$$

  - If $k > j$, $\tilde{R}_{k,t} > \tilde{R}_{j,t}$ if $u'(y_t) < Eu'(y)$
  - Shorter rates are below longer rates if consumption today is relatively high (people want to save for the future).