# ECOM 009 Macroeconomics B

## Lecture 8

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# Plan of this lecture

- To derive a number of tests of the prediction of the consumption-based CAPM.
- To present two important asset pricing puzzles in macro
  - Equity premium puzzle
  - Risk-free rate puzzle
- To discuss the recent empirical evidence.
- To discuss possible solutions to the puzzle.

# The basic pricing equations

$$R_t^{-1} = E_t\left[m_{t+1}\right] \tag{148}$$

$$p_t^j = E_t\left[m_{t+1}(p_{t+1}^j + y_{t+1}^j)\right] \tag{149}$$

- $R_t^{-1}$ is the price of the risk-free asset
- $p_t^j$ is the price of some risky asset $j$ with dividend process $y_{t+1}^j$.
- $m_{t+1} = \beta\frac{u'(c_{t+1}^k)}{u'(c_t^k)}$ is the stochastic discount factor.

# Price implications

$$
\begin{aligned}
p_t^j &= E_t m_{t+1} E_t(p_{t+1}^j + y_{t+1}^j) + cov_t\left(m_{t+1}, (p_{t+1} + y_{t+1}\right) \\
&= R_t^{-1} E_t(p_{t+1}^j + y_{t+1}^j) + cov_t\left(m_{t+1}, (p_{t+1} + y_{t+1}\right)
\end{aligned}
$$

- Price equals PDV of expected payoff adjusted for risk
- Price is above (below) PDV of expected payoff if payoff is positively (negatively) correlated with the SDF.

# The basic pricing equations in terms of returns

- Let $R^j_{t+1} = (p^j_{t+1} + y^j_{t+1})/p_t$. The basic pricing equations

$$p^j_t = E_t \left[ m_{t+1}(p^j_{t+1} + y^j_{t+1}) \right]$$
$$R_t^{-1} = E_t \left[ m_{t+1} \right]$$

can be rewritten in terms of rates of return as

$$1 = E_t \left[ m_{t+1} R^j_{t+1} \right] \quad (150)$$
$$1 = E_t \left[ m_{t+1} R_t \right] \quad (151)$$

- Subtracting the two equations one obtains

$$E_t \left[ m_{t+1}(R^j_{t+1} - R_t) \right] = 0 \quad (152)$$

- Equations (150)-(152) are the basis for all empirical tests.

# Implications for returns: the risk premium

Equation (152) can be rewritten as

$$0 = E_t m_{t+1} E_t(R^j_{t+1} - R_t) + cov_t\left[m_{t+1}, (R^j_{t+1} - R_t)\right]$$

or

$$E_t(R^j_{t+1} - R_t) = -\frac{cov_t\left[m_{t+1}, (R^j_{t+1} - R_t)\right]}{E_t m_{t+1}}$$

- $E_t R^j_t$ = risk-free rate + risk-adjustment

# Volatility bounds (I): Hansen-Jagannathan

The previous equation holds also unconditionally

$$E(R^j_{t+1} - R_t) = -\frac{cov\left[m_{t+1}, (R^j_{t+1} - R_t)\right]}{Em_{t+1}}$$

- Given the identity $\rho_{xy} = \frac{cov(xy)}{\sigma_x \sigma_y}$ one can write

$$E(R^j_{t+1} - R_t) = -\frac{\rho_{mR^j}\sigma_m \sigma_{R^j}}{Em_{t+1}}$$

- As $|\rho_{mR^j}| \leq 1$ the above equation implies a bound

$$\left|\frac{E(R^j_{t+1} - R_t)}{\sigma_{R^j}}\right| \leq \frac{\sigma_m}{Em_{t+1}}$$

# Hansen-Jagannathan bounds

Two ways to use the previous equation.

- It has to hold for any asset. Hence

$$\max_j \left| \frac{E(R^j_{t+1} - R_t)}{\sigma_{R^j}} \right| \leq \frac{\sigma_m}{Em_{t+1}}$$

  - It provides a lower bound for the coefficient of variation of the SDF.
  - Only SDFs that satisfy the bound are supported by the data.

- Alternatively, given a stochastic discount factor asset returns have to satisfy

$$\begin{cases} E(R^j_{t+1} - R_t) < \frac{\sigma_m}{Em_{t+1}} \sigma_{R^j} & \text{if } E(R^j_{t+1} - R_t) > 0 \\ E(R^j_{t+1} - R_t) > -\frac{\sigma_m}{Em_{t+1}} \sigma_{R^j} & \text{if } E(R^j_{t+1} - R_t) < 0 \end{cases}$$

# An alternative manipulation

Let us use equations (150)-(151) instead to write

$$1 = E_t[e^{\log m_{t+1} + \log R_t}]$$
$$1 = E_t[e^{\log m_{t+1} + \log R^j_{t+1}}]$$

- Assume $\log m_{t+1}$ and $\log R^j_{t+1}$ are jointly normally distributed and homoschedastic (no longer general equilibrium).
- We can write

$$1 = e^{E_t \log m_{t+1} + \log R_t + \frac{var(\log m_{t+1})}{2}}$$

$$1 = e^{E_t \log m_{t+1} + E_t \log R^j_{t+1} + \frac{var(\log m_{t+1})}{2} + \frac{var(\log R^j_{t+1})}{2} + cov(\log m_{t+1}, R^j_{t+1})}$$

# An alternative manipulation II

By taking logs of the last two equations and subtracting the first one from the second one we obtain

$$\log R_t = -E_t \log m_{t+1} - \frac{var(\log m_{t+1})}{2}$$

$$E_t \log R^j_{t+1} - \log R_t + \frac{var(\log R^j_{t+1})}{2} = -cov(\log m_{t+1}, R^j_{t+1})$$

# Parametric restriction: CRRA preferences

- To make progress we need to impose some parametric restriction.
  - $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, which implies
  - $\log m_{t+1} = \log\left[\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\right] = \log\beta - \gamma\Delta\log(c_{t+1})$
- The two equations in the previous slide become

$$\log R_t = -\log\beta + \gamma\Delta\log(c_{t+1}) + \gamma^2\frac{var(\Delta\log(c_{t+1}))}{2}$$

$$E_t\log R^j_{t+1} - \log R_t + \frac{var(\log R^j_{t+1})}{2} = \gamma cov(\Delta\log c_{t+1}, R^j_{t+1})$$

# The basic equations with CRRA preferences

With CRRA preferences equations (151)- (152) can be written as

$$1 = E_t \left[ \beta \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma} R_t \right]$$
$$0 = E_t \left[ \beta \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma} (R^j_{t+1} - R_t) \right]$$

# Testing for pricing errors: averages

One could use the previous two equations to create pricing errors.

$$\epsilon^{d}_{t+1} = \left[\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} R_t\right] - 1$$

$$\epsilon^{e-d} = \left[\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} (R^{j}_{t+1} - R_t)\right]$$

- The first equation relates to the *risk-free rate.*
- The second equation relates to the *equity premium.*
- Under the null hypothesis (which can be tested) the - conditional and unconditional expectation - of the errors have to be zero.

# Equity premium volatility test

$$\log R_t = -\log\beta + \gamma\Delta\log(c_{t+1}) + \gamma^2\frac{var(\Delta\log(c_{t+1}))}{2} \quad (153)$$

$$E_t \log R^j_{t+1} - \log R_t + \frac{var(\log R^j_{t+1})}{2} = \gamma cov(\Delta\log c_{t+1}, R^j_{t+1})$$
$$\leq \gamma\sqrt{var(\log R^j_{t+1}) var(\Delta\log(c_{t+1}))}$$

- The second equation relates the excess return on the risk free asset to the covariance between consumption growth and such return (it is the equation behind the **equity premium volatility puzzle**).
- The first equation relates the risk free rate to consumption growth and its volatility (it is the equation behind the **risk free rate puzzle**)

# The data

- The equations on the previous two slides have been used to test how well the representative agent model can explain asset returns. $R^j_{t+1}$ has traditionally been the return on the S&P stock market index, $R_t$ the return on US T-bills and consumption is average US per capita consumption. Note that the law of iterated expectations implies that we can replace $E_t$ (the conditional expectation operator) by $E_0$ (the unconditional expectation operator); i.e. use time averages.
- Recently, McGrattan and Prescott (2003) have proposed some corrections to the data.

# The data

**Table 1: Means and Standard Errors of Measurements**

| | S&P 500 | Adj. S&P 500 | T-Bill Rate | Long-term Debt Rate |
|---|---|---|---|---|
| mean | 8.30% | 5.08% | 1.32% | 2.90% |
| std | 20.44% | 20.37% | 5.10% | 3.25% |

- The above table and all the following ones come from Imrohoroglu (2009).
- They report all data and tests for the original data (S&P 500 and T-Bill rate) and the modified stock market data (Adj. S&P 500) and modified bond data (Long-term Debt rate) proposed by McGrattan and Prescott (2003).

# Pricing errors: mean of $\epsilon_{t+1}^{e-d}$

**Table 2: Equity Premium Puzzle: mean of $\{\epsilon_{t+1}^{e-d}\}$**

| | S&P500/T-B | | Adj. S&P500/T-B | | S&P 500/Debt | | Adj. S&P500/Debt | |
|---|---|---|---|---|---|---|---|---|
| $\gamma$ | $\overline{\epsilon}$ | t-stat | $\overline{\epsilon}$ | t-stat | $\overline{\epsilon}$ | t-stat | $\overline{\epsilon}$ | t-stat |
| 0 | 0.0698 | 3.73 | 0.0376 | 2.03 | 0.0540 | 2.88 | 0.0218 | 1.17 |
| 1 | 0.0682 | 3.67 | 0.0365 | 1.98 | 0.0527 | 2.82 | 0.0211 | 1.14 |
| 2 | 0.0667 | 3.60 | 0.0355 | 1.93 | 0.0516 | 2.77 | 0.0204 | 1.10 |
| 3 | 0.0653 | 3.52 | 0.0346 | 1.88 | 0.0504 | 2.71 | 0.0197 | 1.07 |
| 4 | 0.0639 | 3.44 | 0.0336 | 1.83 | 0.0494 | 2.66 | 0.0191 | 1.04 |
| 5 | 0.0626 | 3.36 | 0.0327 | 1.77 | 0.0485 | 2.60 | 0.0186 | 1.00 |
| 6 | 0.0614 | 3.27 | 0.0319 | 1.71 | 0.0476 | 2.53 | 0.0181 | 0.97 |
| 7 | 0.0602 | 3.18 | 0.0310 | 1.65 | 0.0468 | 2.47 | 0.0176 | 0.93 |
| 8 | 0.0591 | 3.08 | 0.0302 | 1.58 | 0.0460 | 2.40 | 0.0172 | 0.90 |
| 9 | 0.0580 | 2.98 | 0.0294 | 1.52 | 0.0453 | 2.33 | 0.0167 | 0.87 |
| 10 | 0.0570 | 2.87 | 0.0287 | 1.45 | 0.0447 | 2.26 | 0.0164 | 0.83 |

# Pricing errors: mean of $\epsilon^d_{t+1}$

**Table 3: Low Risk-Free Rate Puzzle: mean of $\{\epsilon^d_{t+1}\}$**

| | T-Bill Yield | | L-T Debt Yield | | L-T Debt Yield, no war | |
|---|---|---|---|---|---|---|
| $\gamma$ | $\overline{\epsilon}$ | t-stat | $\overline{\epsilon}$ | t-stat | $\overline{\epsilon}$ | t-stat |
| 0 | 0.0031 | 0.65 | 0.0187 | 6.18 | 0.0276 | 12.75 |
| 1 | −0.0134 | −2.28 | 0.0019 | 0.42 | 0.0115 | 2.61 |
| 2 | −0.0287 | −3.53 | −0.0137 | −1.96 | −0.0031 | −0.39 |
| 3 | −0.0426 | −3.96 | −0.0280 | −2.86 | −0.0163 | −1.41 |
| 4 | −0.0554 | −4.04 | −0.0410 | −3.24 | −0.0281 | −1.84 |
| 5 | −0.0669 | −4.03 | −0.0529 | −3.38 | −0.0386 | −2.03 |
| 6 | −0.0772 | −3.92 | −0.0636 | −3.40 | −0.0477 | −2.08 |
| 7 | −0.0864 | −3.77 | −0.0731 | −3.35 | −0.0555 | −2.06 |
| 8 | −0.0943 | −3.60 | 0.0814 | −3.25 | −0.0619 | −2.00 |
| 9 | −0.1011 | −3.40 | −0.0886 | −3.11 | −0.0671 | −1.90 |
| 10 | −0.1067 | −3.20 | −0.0946 | −2.96 | −0.0709 | −1.78 |

# The equity premium volatility puzzle

**Table 4: The Equity Premium Puzzle**

| | $\overline{aer_e}$ | $\sigma_{er_m}$ | $\sigma(m)$ | $\sigma_{\Delta c}$ | $corr_{er_m,\Delta c}$ | $cov_{er_m,\Delta c}$ | $RRA(1)$ | $RRA(2)$ |
|---|---|---|---|---|---|---|---|---|
| S&P500/T-B | 6.77 | 19.48 | 34.73 | 3.21 | 0.0917 | 5.74 | 117.79 | 10.80 |
| Adj. S&P500/T-B | 3.73 | 18.82 | 18.66 | 3.21 | 0.0910 | 5.85 | 63.75 | 5.80 |
| S&P500/Debt | 5.18 | 19.65 | 26.37 | 3.21 | 0.0906 | 5.72 | 90.56 | 8.20 |
| Adj. S&P500/Debt | 2.15 | 20.19 | 10.66 | 3.21 | 0.0898 | 5.83 | 36.94 | 3.32 |
| Ignore 1935-1960 | 0.67 | 19.86 | 3.39 | 3.46 | 0.1148 | 7.89 | 8.53 | 0.98 |

$$\underbrace{E_t \log R^j_{t+1} - \log R_t + \frac{var(\log R^j_{t+1})}{2}}_{\overline{aer_e}} = \gamma \underbrace{cov(\Delta \log c_{t+1}, R^j_{t+1})}_{cov_{er_m,\Delta c}/100}$$

# Equity premium volatility puzzle: bottom line

- Given the low covariance between consumption growth and the stock market index the coefficient of risk aversion which reconciles such covariance with the high equity premium must be way too large (this is the **equity premium volatility puzzle** pointed out by Mehra and Prescott 1985).
- The return on the stock market is too high given the (non-diversifiable) risk involved.
  - Low covariance is due to (1) low consumption growth variance; (2) low correlation between consumption growth and stock returns.
  - (1) is the real source problem

# The risk-free rate puzzle

**Table 5: The Riskfree Rate Puzzle**

| | $\overline{r_d}$ | $\overline{\Delta c}$ | $\sigma(\Delta c)$ | $RRA(1)$ | $TPR(1)$ | $RRA(2)$ | $TPR(2)$ |
|---|---|---|---|---|---|---|---|
| S&P500/T-B | 1.20 | 1.79 | 3.21 | 117.79 | 15,770.99 | 10.80 | −11.46 |
| Adj. S&P500/T-B | 1.20 | 1.79 | 3.21 | 63.75 | 163.24 | 5.80 | −7.21 |
| S&P500/Debt | 2.85 | 1.79 | 3.21 | 90.56 | 1302.13 | 8.20 | −8.09 |
| Adj. S&P500/Debt | 2.85 | 1.79 | 3.21 | 36.94 | 7.28 | 3.32 | −2.54 |
| Ignore 1935-1960 | 3.77 | 1.71 | 3.46 | 8.53 | −6.29 | 0.98 | 2.11 |

$$\log R_t = -\log\beta + \gamma\underbrace{\Delta\log(c_{t+1})}_{\Delta c} + \gamma^2 \underbrace{\frac{var(\Delta\log(c_{t+1}))}{2}}_{var(\Delta c)}$$

# Risk-free rate puzzle: bottom line

- *Given the risk aversion coefficient necessary to eliminate the equity premium* the risk free rate is too low (or the subjective discount rate must be negative to fit the data) (this is the **low risk free rate puzzle** pointed out by Weil 1989). Note that it is there only as a consequence of the risk aversion coefficient having to be high enough to fit the equity premium.
- Alternatively, given that consumption per capita grows over time at roughly 3% either the precautionary saving motive is large (and it is not using aggregate data) or consumption tilting requires a high risk free rate relative to the subjective discount rate.

# Assumptions underlying the consumption CAPM

To understand were the problem may lie think about the assumptions underlying the equations.

1. Time separable preferences which associate risk aversion and intertemporal elasticity of substitution.
2. Complete markets. This allowed us to replace the individual stochastic discount factor (a function of individual consumption) with the economy-wide stochastic discount factor (a function of *average per capita consumption*
3. No frictions/transaction costs in trading assets. Otherwise the return on share may partly reflect higher taxes and/or higher transaction costs rather than just risk.

# Possible solutions to the puzzles

1. Non-standard preferences (e.g. Epstein-Zin) decouple risk-aversion from intertemporal elasticity of substitution.
   - Without further restrictions, they can explain the low risk free rate puzzle but not the equity premium one (the risk aversion coefficient still need to be high).
   - Some hope if consumption - i.e. dividend in general equilibrium - *growth* and consumption volatility are a persistent process (roughly AR(1)). See Bansal and Yaron (2004).
2. Incomplete markets are unlikely to take us very far unless they imply very high consumption variability (i.e. income shocks are highly persistent) (think about equation (153)).
3. Transaction costs are dealt with in the tables above. They are not large enough to solve the puzzle.